

EXPLICIT CONSTRUCTIONS OF RAMANUJAN COMPLEXES

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ABSTRACT. In this paper we present for every $d \geq 2$ and every local field F of positive characteristic, explicit constructions of Ramanujan complexes which are quotients of the Bruhat-Tits building $\mathcal{B}_d(F)$ associated with $\mathrm{PGL}_d(F)$.

1. INTRODUCTION

In [LSV] we defined and proved the existence of Ramanujan complexes, see also [B1], [CSZ] and [Li]. The goal of this paper is to present an explicit construction of such complexes.

Our work is based on the lattice constructed by Cartwright and Steger [CS]. This remarkable discrete subgroup Γ of $\mathrm{PGL}_d(F)$, when F is a local field of positive characteristic, acts simply transitively on the vertices of the Bruhat-Tits building $\mathcal{B}_d(F)$, associated with $\mathrm{PGL}_d(F)$. By choosing suitable congruence subgroups of Γ , we are able to present the 1-skeleton of the corresponding finite quotients of $\mathcal{B}_d(F)$ as Cayley graphs of explicit finite groups, with specific sets of generators. The simplicial complex structure is then defined by means of these generators.

Let $\begin{bmatrix} d \\ k \end{bmatrix}_q$ denote the number of subspaces of dimension k of \mathbb{F}_q^d .

Theorem 1.1. *Let q be a prime power, $d \geq 2$, $e \geq 1$ ($e > 1$ if $q = 2$).*

Then, the group $G = \mathrm{PGL}_d(\mathbb{F}_{q^e})$ has an (explicit) set S of $\begin{bmatrix} d \\ 1 \end{bmatrix}_q + \begin{bmatrix} d \\ 2 \end{bmatrix}_q + \cdots + \begin{bmatrix} d \\ d-1 \end{bmatrix}_q$ generators, such that the Cayley complex of G with respect to S is a Ramanujan complex, covered by $\mathcal{B}_d(F)$, when $F = \mathbb{F}_q((y))$.

The Cayley complex of G with respect to a set of generators S is the simplicial complex whose 1-skeleton is the Cayley graph $\mathrm{Cay}(G; S)$, where a subset of $i+1$ vertices is an i -cell iff every two vertices comprise an edge. The generators in Theorem 1.1 are explicitly given in Section 9.

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In the case $d = 2$ there are two types of Ramanujan graphs, bi-partite, and non-bi-partite ([LPS], [Mo], [Lu1]). Here too, given r dividing d , we produce r -partite complexes, by proving an analog of Theorem 1.1 for every subgroup of $\mathrm{PGL}_d(\mathbb{F}_{q^e})$ containing $\mathrm{PSL}_d(\mathbb{F}_{q^e})$. There is also a version of the theorem for subgroups of $\mathrm{PGL}_d(L)$ where L is a finite local ring.

The paper is organized as follows: In Section 2 we describe affine buildings of type A_d in general, in the language of \mathcal{O} -sublattices of F^d (where \mathcal{O} is the valuation ring of F). The Cartwright-Steger group Γ is constructed in Sections 3–4. Our construction slightly differs from theirs, but is more convenient for the computations to follow. The simply transitive action of Γ on the building is used in Section 5 to describe the defining relations of Γ .

In Section 6 we present and analyze the finite Ramanujan quotients. In particular, we use [LSV] to show that the complexes constructed here are Ramanujan. We should note that the proof of this result in [LSV] relies on what is called the global ‘Jacquet-Langlands correspondence’ for function fields (a correspondence between automorphic representations of a division algebra and of GL_d , cf. [HT, Thm VI.1.1] for the characteristic zero case). This correspondence in the function field case is also considered to be true by experts, as the main ingredients of the proof are known; though the task of writing down a complete proof has not been carried out yet.

Originally, Γ is a group of $d^2 \times d^2$ matrices over a ring R . Section 8 provides an explicit embedding of Γ into $d \times d$ matrices over a finite extension of R . In Section 9 this embedding is refined to identify finite quotients of Γ with subgroups of $\mathrm{PGL}_d(L)$ which contain $\mathrm{PSL}_d(L)$, where L is a finite local ring; a detailed algorithm is given, with an example in Section 10. In particular the generators of Γ are given as $d \times d$ matrices over $\mathbb{F}_q[x]$. For the convenience of the reader, we include a glossary in the final section.

Added in Proof: We recently learned that Alireza Sarveniazi [Sa] has also given an explicit construction of Ramanujan complexes.

2. BUILDINGS

To every reductive algebraic group over a local field one can associate a building, which is a certain simplicial complex, on which the group acts (see [R]). This complex plays the role of a symmetric space for Lie groups.

Recall that a complex is a structure composed of i -cells, where the 0-cells are called the vertices, and an i -cell is a set of $i + 1$ vertices.

A complex is simplicial if every subset of a cell is also a cell. The i -skeleton is the set of all i -cells in the complex. Buildings are in fact clique complexes, which means that a set of $i + 1$ vertices is a cell iff every two vertices form a 1-cell. This property holds for quotient complexes, which will be the subject of Section 6.

We will now describe the affine building associated to $\mathrm{PGL}_d(F)$, where F is a local field. These are called ‘buildings of type \tilde{A}_{d-1} ’ because of the Dynkin diagram of the associated Weil group (which is isomorphic to $S_d \ltimes \mathbb{Z}^{d-1}$). Let \mathcal{O} denote the valuation ring of F ; choose a uniformizer ϖ (so for $F = \mathbb{F}_q((y))$, $\mathcal{O} = \mathbb{F}_q[[y]]$ and $\varpi = y$), and assume $\mathcal{O}/\varpi\mathcal{O} = \mathbb{F}_q$. Consider the \mathcal{O} -lattices of full rank in F^d . For every lattice L , ϖL is a sublattice, and as \mathbb{F}_q -vector spaces $L/\varpi L \cong \mathbb{F}_q^d$. We define an equivalence relation by setting $L \sim sL$ for every $s \in F^\times$.

Remark 2.1. Let $\langle \varpi \rangle$ denote the multiplicative subgroup of F^\times generated by ϖ , and let \mathcal{O}^\times be the invertible elements of \mathcal{O} . Then $F^\times = \langle \varpi \rangle \cdot \mathcal{O}^\times$.

Since $L = sL$ for any element $s \in \mathcal{O}^\times$, the equivalence classes have the form $[L] = \{\varpi^i L\}_{i \in \mathbb{Z}}$. Let \mathcal{B}^0 be the graph whose vertices are the equivalence classes. There is an edge from $[L]$ to a class $x \in \mathcal{B}^0$, iff there is a representative $L' \in x$ such that $\varpi L \subset L' \subset L$. Notice that this is a symmetric relation, since then $\varpi L' \subset \varpi L \subset L$.

The vertices of \mathcal{B}^0 form the 0-skeleton of a complex \mathcal{B} , and the edges are the 1-skeleton, \mathcal{B}^1 . As i -cells of \mathcal{B} we take the complete subgraphs of size $i + 1$ of \mathcal{B}^0 , which correspond to flags

$$\varpi L_0 \subset L_i \subset \cdots \subset L_1 \subset L_0;$$

the i -skeleton is denoted \mathcal{B}^i . It immediately follows that \mathcal{B} has $(d - 1)$ -cells (corresponding to maximal flags in quotients $L/\varpi L$). It also follows that there are no higher dimensional cells.

The group $\mathrm{GL}_d(F)$ acts transitively on lattices by its action on bases. moreover note that the action preserves inclusion of lattices. We call $L_0 = \mathcal{O}^d \subseteq F^d$ the *standard lattice*. If $\tau \in \mathrm{GL}_d(F)$ has entries in \mathcal{O} , then $\tau L_0 \subseteq L_0$. The stabilizer of L_0 in G is thus the maximal compact subgroup $\mathrm{GL}_d(\mathcal{O})$. According to the definition of the equivalence relation, the scalar matrices of $\mathrm{GL}_d(F)$ act trivially on \mathcal{B} , so the action of $\mathrm{GL}_d(F)$ induces a well defined action of $\mathrm{PGL}_d(F)$ on (the vertices of) \mathcal{B} , which is easily seen to be an action of an automorphism group. Again, the stabilizer of $[L_0]$ is the maximal compact subgroup $\mathrm{PGL}_d(\mathcal{O})$. The set of vertices can thus be identified with $\mathrm{PGL}_d(F)/\mathrm{PGL}_d(\mathcal{O})$.

Since the only ideals of \mathcal{O} are powers of $\langle \varpi \rangle$, The Invariant Factor Theorem for F asserts that any matrix in $\mathrm{GL}_d(F)$ can be decomposed

as aga' for $a, a' \in \mathrm{GL}_d(\mathcal{O})$ and $g = \mathrm{diag}(\varpi^{i_1}, \dots, \varpi^{i_d})$, where $i_1 \leq \dots \leq i_d$ are integers. If $L = aga'L_0 = agL_0$, then $\varpi^{-i_1}L = a(\varpi^{-i_1}g)L_0 \subseteq L_0$, and on the other hand $\varpi^{-i_d}L = a(\varpi^{-i_d}g)L_0 \supseteq L_0$, so any two lattices of maximal rank are commensurable. Moreover in this case, $L_0/\varpi^{-i_1}L$ is annihilated by $\varpi^{i_d-i_1}$, and so is a module over $\mathcal{O}/\varpi^{i_d-i_1}\mathcal{O}$, a local ring of order $q^{i_d-i_1}$. In particular $[L_0:\varpi^{-i_1}L]$ is a q power.

This basic fact allows us to define a color function $\varrho:\mathcal{B}^0 \rightarrow \mathbb{Z}/d$ by $\varrho(L) = \log_q [L_0:\varpi^i L]$ for large enough i ; this function is well defined since $[\varpi^i L:\varpi^{i+1}L] = q^d$. Also notice that $\varrho([\tau L_0]) = \nu_0(\det(\tau)) \pmod{d}$, when ν_0 is the valuation of F . This shows that $\mathrm{SL}_d(F)$ is color preserving, while on the other hand, $\tau_\varpi = \mathrm{diag}(\varpi, 1, \dots, 1)$ has determinant ϖ , so $\varrho(\tau_\varpi(L)) = \varrho(L) + 1$ for every L . It follows that $\mathrm{GL}_d(F)$ acts transitively on colors.

The color is additive in the sense that if $L'' \subseteq L'$, then $\varrho(L'') = \varrho(L') + \log_q [L':L'']$. Similarly if $L \subseteq L_0$ and $\tau \in \mathrm{GL}_d(F)$, then $\varrho(\tau L) = \varrho(\tau L_0) + \varrho(L)$.

The colors provide us with d Hecke operators, defined on functions of \mathcal{B}^0 by summation over the neighbors of fixed color-shift:

$$A_k f(x) = \sum_{y \sim x, \varrho(y) - \varrho(x) \equiv k} f(y).$$

These operators generate the Hecke algebra $H(\mathrm{PGL}_d(F), \mathrm{PGL}_d(\mathcal{O}))$ (see [LSV, Sec. 2] for more details).

For $1 \leq t < d$, let $\mathcal{B}[t]$ denote the graph defined on the vertices \mathcal{B}^0 , with the edges $(x, x') \in \mathcal{B}^1$ for which there are $L \in x$ and $L' \in x'$ such that $L' \subseteq L$ and $[L:L'] = q^t$ (in particular, $\varrho(x) - \varrho(x') = t$).

Remark 2.2. *If $L' \subset L$ is a sublattice of index q , then $[L], [L']$ are connected in \mathcal{B}^1 .*

Proof. We need to prove that $\varpi L \subseteq L'$, but this is obvious since L/L' is annihilated (as an \mathcal{O} -module) by multiplication by ϖ . \square

For every $L' \subseteq L$ there is a composition series of sublattices $L' = L^{(m)} \subseteq L^{(m-1)} \subseteq \dots \subseteq L^{(0)} = L$, such that $[L^{(i)}:L^{(i+1)}] = q$. It follows that $\mathcal{B}[1]$ is a connected (directed) subgraph of \mathcal{B} . In fact, $\mathcal{B}[1]$ determines \mathcal{B}^1 , and thus all of \mathcal{B} :

Proposition 2.3. *Vertices x, x' are connected in \mathcal{B}^1 iff there is a chain $x_0, x_1, \dots, x_d \in \mathcal{B}^0$ such that $x_0 = x_d = x$, $(x_i, x_{i+1}) \in \mathcal{B}[1]$ for $i = 0, \dots, d-1$, and such that $x' \in \{x_1, \dots, x_{d-1}\}$.*

Proof. First assume that such a chain exists, and choose representatives $L_i \in x_i$ with $L_d = \varpi L_0$. By the definition of $\mathcal{B}[1]$, we may assume that

$[L_i : L_{i+1}] = q$ and then $L_d \subset \cdots \subset L_2 \subset L_1 \subset L_0$. In particular $([L_i], [L_0]) \in \mathcal{B}^1$ for every i .

On the other hand, if

$$(1) \quad \varpi L_0 \subset L' \subset L_0$$

are lattices, we can lift a maximal flag in $L/\varpi L_0 \cong \mathbb{F}_q^d$ to a maximal chain of lattices refining (1), resulting in a chain x_0, \dots, x_d . \square

Corollary 2.4. *If $(x, x') \in \mathcal{B}[t]$, then there is a path of length t in $\mathcal{B}[1]$ from x to x' .*

Using this criterion, it is easy to see that if the greatest common divisor (t', d) equals t , and $(x, x') \in \mathcal{B}[t']$, then there is a path from x to x' in $\mathcal{B}[t]$. In particular, $\mathcal{B}[t']$ has the same connected components as $\mathcal{B}[t]$. The final result of this section is not needed in the rest of the paper.

We thank the referee for some simplification in the proof of the following proposition.

Proposition 2.5. *Let t be a divisor of d . If t divides $\varrho(x') - \varrho(x)$, then there is a path from x to x' in the (directed) graph $\mathcal{B}[t]$ defined above.*

Proof. We may assume $x = [L_0]$, and $x' = [L]$ with $L = aga'L_0 = agL_0$, $a, a' \in \mathrm{GL}_d(\mathcal{O})$ and $g = \mathrm{diag}(\varpi^{i_1}, \dots, \varpi^{i_d})$. We may assume $0 \leq i_1 \leq \cdots \leq i_d$, and in particular $L \subseteq L_0$. If the claim is true for $gL_0 \subseteq L_0$, then acting with a we get a chain from $aL_0 = L_0$ to L ; we can thus assume $a = 1$.

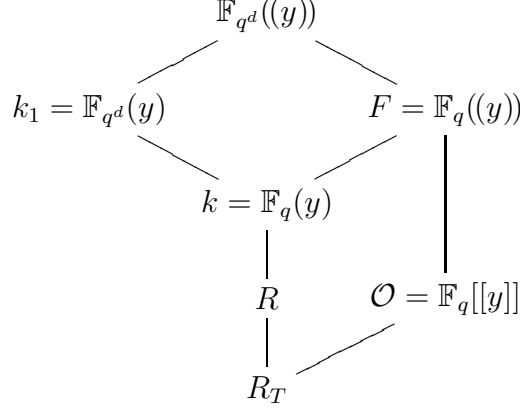
Before going on by induction, we multiply L by a power of ϖ so that

$$(2) \quad i_1 \geq \frac{t}{d-t}(i_d - i_1).$$

Now, by assumption, $i_1 + \cdots + i_d = rt$ for some $r \in \mathbb{N}$.

Case 1. If $i_{d-t+1} > i_1$, then we can lower t of the entries i_2, \dots, i_d , each by 1, keeping the increasing order. Let i'_1, \dots, i'_d denote the resulting values and $g' = \mathrm{diag}(\varpi^{i'_1}, \dots, \varpi^{i'_d})$. Then $gL_0 \subset g'L_0 \subseteq L_0$ with $[g'L_0 : gL_0] = q^t$ and $\varpi g'L_0 \subseteq gL_0$, so $(g'L_0, gL_0) \in \mathcal{B}[t]$. Since the condition $i_1 \geq \frac{d}{t}(i_d - i_1)$ still holds (as i_1 was not changed), we are done by induction on r .

Case 2. Now assume $i_{d-t+1} = i_1$; let j be maximal with $i_j = i_1$. If $j = d$ then $i_1 = \cdots = i_d$ so $L \equiv L_0$ and we are done. Therefore assume $j < d$. Of course $j \geq d - t + 1$. If $i_1 = 0$ then $i_d = 0$ too by the assumption (2), so $L = L_0$ and again we are done. Assume $i_1 > 0$ and $j < d$. In the first step we can lower i_{j+1}, \dots, i_d , but we also have to lower $i_1, \dots, i_{t-(d-j)}$ in order to change exactly t components. We continue lowering the highest entries, using the remaining $d - t$ entries

FIGURE 1. Subrings of $F_{q^d}((y))$

$i_{t-(d-j)+1}, \dots, i_j$ whenever necessary. In each of the next $d/t - 1$ steps, i_d is lowered by one with i_1 unharmed—so at the end our condition (2) is still met. The d/t modules we constructed form a chain in L_0 , climbing from L using $\mathcal{B}[t]$ steps, and ending with indices which now sum up to $rt - d$, so again we are done by induction. \square

Vertices x and x' for which (t, d) does not divide $\varrho(x') - \varrho(x)$ cannot be connected in $\mathcal{B}[t]$, so we proved that $\mathcal{B}[t]$ has (t, d) connected components for every $1 \leq t < d$.

3. THE ARITHMETIC LATTICE

Let \mathbb{F}_q denote the field of order q (a prime power), and \mathbb{F}_{q^d} the extension of dimension d . Let ϕ denote a generator of the Galois group $\text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$. Fix a basis $\zeta_0, \dots, \zeta_{d-1}$ for \mathbb{F}_{q^d} over \mathbb{F}_q , where $\zeta_i = \phi^i(\zeta_0)$.

Extend ϕ to an automorphism of the function field $k_1 = \mathbb{F}_{q^d}(y)$ by setting $\phi(y) = y$; the fixed subfield is $k = \mathbb{F}_q(y)$, of co-dimension d . Let ν_y denote the valuation defined by $\nu_y(a_m y^m + \dots + a_n y^n) = m$ ($a_m \neq 0$, $m < n$), and set $F = \mathbb{F}_q((y))$, the completion with respect to ν_y , and $\mathcal{O} = \mathbb{F}_q[[y]]$, its ring of integers.

Let

$$R = \mathbb{F}_q \left[y, \frac{1}{y}, \frac{1}{1+y} \right] \subseteq k,$$

and let R_T denote the subring

$$(3) \quad R_T = \mathbb{F}_q \left[y, \frac{1}{1+y} \right].$$

Since $1+y$ is invertible in \mathcal{O} , $R_T \subseteq \mathcal{O}$.

For a commutative R_T -algebra S (namely a commutative ring with unit which is an R_T -module, e.g. k , F , R or R/I for an ideal $I \triangleleft R$), we denote by y the element $y \cdot 1 \in S$. For such S we define an S -algebra $\mathcal{A}(S)$, by

$$\mathcal{A}(S) = \bigoplus_{i,j=0}^{d-1} S \zeta_i z^j,$$

with the relations

$$(4) \quad z \zeta_i = \phi(\zeta_i) z, \quad z^d = 1 + y.$$

The center of $\mathcal{A}(S)$ is S . We will frequently use the fact that for an R_T -algebra S , $\mathcal{A}(S) = \mathcal{A}(R_T) \otimes_{R_T} S$. It is well known that $\mathcal{A}(k)$ is a central simple algebra, and so there is a norm map $\mathcal{A}(k)^\times \rightarrow k^\times$, which induces a norm map $\mathcal{A}(S)^\times \rightarrow S^\times$ for every $S \subseteq k$. The norm map is a homogeneous form of degree d (in the coefficients of the basis elements $\{\zeta_i z^j\}$), so the norm is also defined for quotients $\mathcal{A}(S/I)$. We remark that $\mathbb{F}_{q^d} \otimes_{\mathbb{F}_q} S$ is Galois over S and $1 + y$ is invertible in S , so $\mathcal{A}(S)$ is an Azumaya algebra over S (see [DI], where they are called ‘central separable algebras’). This fact will not be used in the rest of the paper.

If $\mathcal{A}(S) \cong M_d(S)$, we say that $\mathcal{A}(S)$ is *split*. We need a criterion for this to happen. If $S_1 = \mathbb{F}_{q^d} \otimes_{\mathbb{F}_q} S$ is a field (so necessarily S is a subfield), then $\mathcal{A}(S)$ is the cyclic algebra $(S_1/S, \phi, 1+y) = S_1[z]$ with the relations in (4). This is a simple algebra of degree d over its center S . Recall Wedderburn’s norm criterion for cyclic algebras [Jac, Cor. 1.7.5]: the algebra $\mathcal{A}(S) = (S_1/S, \phi, 1+y)$ splits iff $1+y$ is a norm in the field extension S_1/S . More generally, the exponent of $(S_1/S, \phi, 1+y)$ (i.e. its order in the Brauer group $\text{Br}(S)$) is the minimal $i > 0$ such that $(1+y)^i$ is a norm. In particular, if this exponent is $d = [S_1:S]$, $\mathcal{A}(S)$ is a division algebra (since the exponent of a central simple algebra is always bounded by the degree of the underlying division algebra).

The algebra $\mathcal{A}(R)$ will later be used to construct the desired complexes. As mentioned above, $\mathcal{A}(R) \subseteq \mathcal{A}(k)$, k being the ring of fractions of R ; moreover, $\mathcal{A}(k)$ is the ring of central fractions of this algebra,

$$\mathcal{A}(k) = (R - \{0\})^{-1} \mathcal{A}(R).$$

We now consider completions of $\mathcal{A}(k)$. The global field $k = \mathbb{F}_q(y)$ has the minus degree valuation, defined for $f, g \in \mathbb{F}_q[y]$ by $\nu(f/g) = \deg(g) - \deg(f)$. Also recall that the other nonarchimedean discrete valuations of $\mathbb{F}_q(y)$ are in natural correspondence with the prime polynomials of $\mathbb{F}_q[y]$. For a prime polynomial $p \in \mathbb{F}_q[y]$, the valuation is defined by $\nu_p(p^i f/g) = i$, where f, g are polynomials prime to p . The ring of p -adic integers in k is $\mathbb{F}_q[y]_p = \{f/g : (p, g) = 1\}$. The completion

$$\begin{array}{ccc}
\mathrm{GL}_d(F) & \xrightarrow{\det} & F^\times \\
\uparrow & & \uparrow \\
\mathcal{A}(R)^\times & \xrightarrow{N} & R^\times
\end{array}$$

FIGURE 2. Norm and determinant

with respect to a valuation $\nu = \nu_p$ is $k_\nu = \mathbb{F}_q[y]_p((p)) = \{\sum_{i=-v}^{\infty} \alpha_i p^i\}$ (which we will also denote by k_p). The ring of integers of k_p is $\mathcal{O}_p = \mathbb{F}_q[y]_p[[p]]$. The notation $\nu_{1/y}$ is used for the degree valuation, since the completion of k with respect to this valuation is $\mathbb{F}_q((1/y))$; moreover, filtration by the ideal $\langle 1/y \rangle$ of the ring of integers $\mathbb{F}_q[[1/y]]$ determines the valuation.

The Albert-Brauer-Hasse-Noether Theorem describes division algebras over k in terms of their local invariants, which translates to an injection $\mathrm{Br}(k) \rightarrow \oplus \mathrm{Br}(k_p)$. More precisely, the d -torsion part of $\mathrm{Br}(k_p)$ is cyclic of order d for every valuation ν_p . Taking the unramified extension k'_p/k_p of dimension d , these d classes can be written as the cyclic algebras $(k'_p/k_p, \phi, \varpi^i)$ for $i = 0, \dots, d-1$, where ϖ is a uniformizer. If $(k_1/k, \phi, c)$ is a cyclic k -algebra, the local invariants are determined by the values $\nu(c)$ ([P, Chaps. 17–18] is a standard reference, though the focus is on number fields).

There are only two valuations ν of k for which $\nu(1+y) \neq 0$, namely ν_{1+y} and $\nu_{1/y}$, for which the values are 1 and -1 , respectively. We thus have

Proposition 3.1. *The completions $\mathcal{A}(\mathbb{F}_q((1/y)))$ and $\mathcal{A}(\mathbb{F}_q((1+y)))$ of $\mathcal{A}(k)$ are division algebras. On the other hand, for any other completion k_ν of k , $\mathcal{A}(k_\nu)$ splits.*

In particular

$$(5) \quad \mathcal{A}(R) \subseteq \mathcal{A}(k) \otimes_k F = \mathcal{A}(F) \cong M_d(F).$$

The same argument embeds

$$(6) \quad \mathcal{A}(R_T) \subseteq \mathcal{A}(\mathcal{O}) \cong M_d(\mathcal{O}).$$

We use the algebras $\mathcal{A}(S)$ to define algebraic group schemes. For an R_T -algebra S , let $\tilde{G}'(S) = \mathcal{A}(S)^\times$, the invertible elements of $\mathcal{A}(S)$, and $G'(S) = \mathcal{A}(S)^\times / S^\times$. Recall that for every R_T -algebra S , one can define a multiplicative function $\mathcal{A}(S) \rightarrow S^\times$ called the reduced norm (e.g. by taking the determinant in a splitting extension of S). In particular, the diagram in Figure 2 commutes. We can thus define $\tilde{G}'_1(S)$ as the set of

elements of $\tilde{G}'(S)$ of norm 1, and $G'_1(S)$ as the image of $\tilde{G}'_1(S)$ under the map $\tilde{G}'(S) \rightarrow G'(S)$ (see the square in the middle of Figure 5).

Remark 3.2. *The sequence*

$$1 \longrightarrow \mu_d(S) \longrightarrow \tilde{G}'_1(S) \longrightarrow G'_1(S) \longrightarrow 1$$

is exact, where $\mu_d(S)$ is the group of d -roots of unity in S .

These group schemes are forms of the classical groups $\tilde{G}(S) = \mathrm{GL}_d(S)$, $G(S) = \mathrm{PGL}_d(S)$, $\tilde{G}_1(S) = \mathrm{SL}_d(S)$ and $G_1(S) = \mathrm{PSL}_d(S)$. If $\mathcal{A}(S)$ is a matrix ring, we have that $\tilde{G}'(S) = \tilde{G}(S)$ and $G'(S) = G(S)$.

It is useful to have equivalent definitions for the groups $G'(S)$ for various rings S . Fix the ordered basis

$$\{\zeta_0, \dots, \zeta_{d-1}, \zeta_0 z, \dots, \zeta_{d-1} z, \dots, \zeta_0 z^{d-1}, \dots, \zeta_{d-1} z^{d-1}\}$$

of $\mathcal{A}(S)$ over S . Conjugation by an invertible element $a \in \mathcal{A}(S)$ is a linear transformation of the algebra. Let $i_S: G'(S) \rightarrow \mathrm{GL}_{d^2}(S)$ be the induced embedding. If $S \subseteq S'$, then the diagram

$$\begin{array}{ccc} G'(S') & \xrightarrow{i_{S'}} & \mathrm{GL}_{d^2}(S') \\ \uparrow & & \uparrow \\ G'(S) & \xrightarrow{i_S} & \mathrm{GL}_{d^2}(S) \end{array}$$

commutes.

Proposition 3.3. *Let $R_T \subseteq S \subseteq S'$ be commutative rings, such that S is a Noetherian unique factorization domain. Then*

$$i_S G'(S) = i_{S'} G'(S') \cap \mathrm{GL}_{d^2}(S),$$

the intersection taken in $\mathrm{GL}_{d^2}(S')$.

Proof. The inclusion $i_S G'(S) \subseteq i_{S'} G'(S') \cap \mathrm{GL}_{d^2}(S)$ is trivial since $\mathcal{A}(S) \subseteq \mathcal{A}(S')$. Let $\alpha \in i_{S'} G'(S') \cap \mathrm{GL}_{d^2}(S)$, then α is an isomorphism of algebras (since it is induced by an element of $\mathcal{A}(S')$) and preserves S (as it belongs to $\mathrm{GL}_{d^2}(S)$). It is thus an automorphism of $\mathcal{A}(S)$, which must be inner [AG, Thm. 3.6]. \square

The proposition covers, in particular, $S = R_T, R, k, k_\nu, \mathcal{O}_\nu$, as well as \bar{R}_T , and \bar{R} which are defined in Section 8, with an arbitrary extension S' (usually taken to be from the same list).

Proposition 3.4. *$G'(R)$ is a discrete subgroup of $G(F)$.*

Proof. The ring $R = \mathbb{F}_q[\frac{1}{y}, y, \frac{1}{1+y}]$ embeds (diagonally) as a discrete subgroup of the product

$$F \times k_{1/y} \times k_p.$$

This can be seen by letting $a_n = \frac{f_n(y)}{y^{i_n}(1+y)^{j_n}}$ be a sequence of non-zero elements in R (with $f_n(\lambda) \in \mathbb{F}_q[\lambda]$ and $i_n, j_n \geq 0$) such that $a_n \rightarrow 0$. We then have that $\nu_y(a_n) \rightarrow \infty$, which implies $i_n = 0$ for n large enough and $\nu_y(f_n(y)) \rightarrow \infty$. Likewise, $\nu_p(a_n) \rightarrow \infty$, so $j_n = 0$ for n large enough. This implies that $\nu_{1/y}(a_n) \rightarrow -\infty$, which is a contradiction.

It follows that the diagonal embedding of $G'(R)$ into

$$G'(F) \times G'(k_{1/y}) \times G'(k_p)$$

is discrete. But an algebraic group over a local field is compact iff it has rank zero [PR]. Therefore, by Proposition 3.1, $G'(k_{1/y})$ and $G'(k_p)$ are compact, and $G'(R)$ is discrete in the other component $G'(F) = G(F)$. \square

In fact, from general results it follows that $G'(R)$ is a cocompact lattice in $G'(F)$, but we will show it directly when we demonstrate that $G'(R)$ acts transitively on the vertices of the affine building of $G(F) = \mathrm{PGL}_d(F)$.

Consider $G(\mathcal{O}) = \mathrm{PGL}_d(\mathcal{O})$, a maximal compact subgroup of $G(F)$, which is equal to $G'(\mathcal{O})$ by Equation (6). Viewing $K = i_{\mathcal{O}}G'(\mathcal{O})$ and $i_R G'(R)$ as subgroups of $i_F G'(F) \subseteq \mathrm{GL}_{d^2}(F)$, the intersection

$$(7) \quad i_{R_T} G'(R_T) = K \cap i_R G'(R)$$

is finite, being the intersection of discrete and compact subgroups (note that $R \cap \mathcal{O} = R_T$).

Proposition 3.5. *$G'(R_T)$ is a semidirect product of $\langle z \rangle \cong \mathbb{Z}/d\mathbb{Z}$ acting on $\mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times$.*

Proof. Recall that $R_T = \mathbb{F}_q[y, 1/(1+y)]$, so that $\mathcal{A}(R_T) = \mathbb{F}_{q^d}[y, 1/(1+y), z]$ with the relations $z\alpha z^{-1} = \phi(\alpha)$ ($\alpha \in \mathbb{F}_{q^d}$) and $z^d = 1+y$. Setting $y = z^d - 1$, we see that $\mathcal{A}(R_T) = \mathbb{F}_{q^d}[z, z^{-1}]$ is a skew polynomial ring with one invertible variable over \mathbb{F}_{q^d} . Every element of $\mathcal{A}(R_T)$ has a monomial αz^r ($\alpha \in \mathbb{F}_{q^d}^\times$) with r maximal, called the upper monomial (with respect to z), and similarly every element has a lower monomial. The upper monomial of a product fg is equal to the product of the respective upper monomial, and likewise for the lower monomials.

Now let $f, g \in \mathcal{A}(R_T)$ be elements with $fg = 1$, then the product of the upper monomials and that of the lower monomials are both equal to 1, proving that f and g are monomials. Thus, the invertible elements

of $\mathcal{A}(R_T)$ are $\tilde{G}(R_T) = \{\alpha z^i : \alpha \in \mathbb{F}_{q^d}^\times, i \in \mathbb{Z}\}$. The result is obtained by taking this modulo the center. \square

4. A SIMPLY TRANSITIVE ACTION ON \mathcal{B}

We continue with the notation of the last section. The embedding (5) of $\mathcal{A}(R)$ into $M_d(F)$ induces embeddings $\tilde{G}'(R) \hookrightarrow \mathrm{GL}_d(F)$ and $G'(R) \hookrightarrow \mathrm{PGL}_d(F)$, so $G'(R)$ acts on \mathcal{B}^0 . Notice that $G'(k)$ is dense in $G'(F) = G(F)$, so its action on \mathcal{B}^0 is transitive.

In [CS], Cartwright and Steger present a subgroup of $G'(k)$, which acts simply transitively on the vertices of the building \mathcal{B} . Identifying \mathcal{B}^0 with a group, when possible, is an important tool in the description of finite quotients of \mathcal{B} . We will use the Cartwright-Steger group, constructed in a different way. In particular we construct the group as a subgroup of $G'(R)$, which is easily shown to be discrete. The proof that it acts transitively relies on the existence of an element of the appropriate norm, and that the action is simple is seen by a relatively easy computation of the group scheme over various rings.

Recall that $\mathcal{A}(R) = R[\zeta_i, z]$ where $z\zeta_i z^{-1} = \phi(\zeta_i)$ and $z^d = 1 + y$, and let

$$b = 1 - z^{-1} \in \mathcal{A}(R).$$

Proposition 4.1. *The reduced norm of b in $\mathcal{A}(R)$ is $y/(1 + y)$.*

Proof. Recall that $\mathcal{A}(R) \subseteq \mathcal{A}(k)$ and that the restriction of the norm function of $\mathcal{A}(k)$ to $\mathcal{A}(R)$ is the norm function of $\mathcal{A}(R)$. As is the case for matrices, if $a \in \mathcal{A}(R)$ generates a subalgebra of dimension d , its norm $N(a)$ is the free coefficient in the characteristic polynomial of a multiplied by $(-1)^d$.

The minimal polynomial of z is $\lambda^d - (1 + y)$, so that $N(z) = (-1)^{d-1}(1 + y)$. Likewise, the minimal polynomial of $z - 1$ is $(\lambda + 1)^d - (1 + y)$, with the free coefficient $-y$, so that $N(z - 1) = (-1)^{d-1}y$. Finally, since $b = z^{-1}(z - 1)$, $N(b) = N(z - 1)/N(z) = y/(1 + y)$. \square

Since $1 + y$ is invertible in \mathcal{O} , we claim

Corollary 4.2. $N(b) \equiv y \pmod{\mathcal{O}^\times}$.

Under the embedding (6), $b \in \mathcal{A}(R_T) \subseteq M_d(\mathcal{O})$, so $bL_0 \subseteq L_0$, where $L_0 = \mathcal{O}^d$ is the standard lattice. By the corollary, and since the diagram in Figure 2 commutes, we have $[L_0 : bL_0] = q$. Note that b is not invertible in $\mathcal{A}(R_T)$ (since the norm $y/(1 + y)$ is not invertible in R_T), but b is invertible in $\mathcal{A}(R)$.

Let Ω denote the set of neighbors of $[L_0]$ in \mathcal{B} which have color 1; they correspond to sublattices of index q in L_0 . Consider the element

$$b_u = ubu^{-1} = 1 - \frac{u}{\phi(u)}z^{-1}$$

where $u \in \mathbb{F}_{q^d}^\times \subseteq \mathcal{A}(R)^\times$. The embedding $\mathcal{A}(R) \subseteq M_d(F)$ extends the regular embedding $\mathbb{F}_{q^d} \hookrightarrow M_d(\mathbb{F}_q)$ (via the basis $\zeta_0, \dots, \zeta_{d-1}$), and in particular it takes u to a matrix with coefficients in \mathbb{F}_q . This proves that $uL_0 = L_0$, and since $\mathrm{GL}_d(F)$ preserves the structure of \mathcal{B} , u permutes the vertices in Ω . Moreover, the sublattices of index q of L_0 correspond to the points of the projective space $\mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times$ via the isomorphism $L_0/yL_0 \cong \mathbb{F}_{q^d}$. The action of u on the latter space is by multiplication, so $\mathbb{F}_{q^d}^\times \subseteq \mathcal{A}(R)^\times$ acts transitively on Ω . Finally, we have that $b_u(L_0) = ubu^{-1}L_0 = u(bL_0)$, so we proved:

Proposition 4.3. *For every lattice $L_1 \in \Omega$, there is an element $u \in \mathbb{F}_{q^d}^\times$ (unique up to multiplication by \mathbb{F}_q^\times) such that $b_u L_0 = L_1$.*

Definition 4.4. *Let $\tilde{\Gamma}$ denote the subgroup*

$$(8) \quad \tilde{\Gamma} = \langle b_u : u \in \mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times \rangle$$

of $\tilde{G}'(R) = \mathcal{A}(R)^\times$, and let Γ be the image of $\tilde{\Gamma}$ under the projection $\tilde{G}'(R) \rightarrow G'(R)$.

Proposition 4.5. *Γ acts transitively on \mathcal{B}^0 .*

Proof. We show that L_0 can be taken to any of its sublattices L by an element of Γ . The proof is by induction on $[L_0:L]$. If $[L_0:L] = q$, then $L \in \Omega$ and $b_u L_0 = L$ for some u . Assume $[L_0:L] > q$, and let $L \subseteq L' \subseteq L_0$ be an intermediate sublattice such that $[L':L] = q$. By the induction hypothesis there is an element $c \in \Gamma$ such that $cL_0 = L'$. The lattices in $c\Omega$ are the sublattices of index q of L' , so for some $u \in \mathbb{F}_{q^d}^\times$, we have that $c(b_u L_0) = L$, which proves our claim. \square

Corollary 4.6. *If $L \subseteq L_0$ and $[L_0:L] = q^\ell$, then there are $u_1, \dots, u_\ell \in \mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times$ such that $[L] = b_{u_1} \dots b_{u_\ell}[L_0]$.*

Moreover, if $L \subseteq L' \subseteq L_0$ where $[L_0:L'] = q^k$ and $[L'] = b_{u_1} \dots b_{u_k}[L_0]$, then for suitable u_{k+1}, \dots, u_ℓ we have $[L] = b_{u_1} \dots b_{u_k} b_{u_{k+1}} \dots b_{u_\ell}[L_0]$.

Let $R_0 = \mathbb{F}_q[1/y]$. As R_0 does not contain R_T , $\mathcal{A}(R_0)$ is not defined, so we cannot define $G'(R_0)$ in the usual way. However, following Proposition 3.3, we set $G'(R_0) = i_k G'(k) \cap \mathrm{GL}_{d^2}(R_0)$.

As mentioned in the introduction, Cartwright and Steger present in [CS] a group Γ' which acts simply transitively on \mathcal{B}^0 . Their group is defined as follows:

Definition 4.7. *The group Γ' is composed of the elements of $G'(R_0)$ which modulo $1/y$ are upper triangular with identity $d \times d$ blocks on the diagonal.*

A careful computation, using $b_u^{-1} = 1 + \frac{1}{y} \sum_{i=0}^{d-1} u \phi^i(u)^{-1} z^i$, reveals that

$$(9) \quad b_u(\zeta z^k) b_u^{-1} = \zeta z^k + f_{u,k}(\zeta) \sum_{i=0}^{k-1} \frac{u}{\phi^i(u)} z^i + \frac{1}{y} f_{u,k}(\zeta) \sum_{i=0}^{d-1} \frac{u}{\phi^i(u)} z^i$$

for every $0 \leq k < d$ and $\zeta \in \mathbb{F}_{q^d}$, where $f_{u,k}(\zeta) = \frac{\phi^k u}{u} \zeta - \frac{\phi^{k-1} u}{\phi^{-1} u} \phi^{-1} \zeta$.

It follows that $b_u \in \Gamma'$, so $\Gamma = \langle b_u \rangle \subseteq \Gamma'$. This leads to an easy proof of the main property of Γ :

Proposition 4.8. *The group Γ acts simply transitively on \mathcal{B}^0 , namely, for $K = G(\mathcal{O})$,*

$$(10) \quad \Gamma \cdot K = G(F),$$

$$(11) \quad \Gamma \cap K = 1.$$

In particular, $\Gamma = \Gamma'$.

Proof. The first equation, equivalent to Γ acting transitively on $\mathcal{B}^0 = G(F)/K$, was proved in Proposition 4.5. To compute the intersection, note that $\Gamma \cap K \subseteq \Gamma' \cap K \subseteq G'(R) \cap K = G'(R_T)$ by Equation (7), so $\Gamma \cap K \subseteq G'(R_T) \cap \Gamma'$. Let ζz^j be an element of $G'(R_T) \cap \Gamma'$ for $\zeta \in \mathbb{F}_{q^d}^\times$ and $j = 0, \dots, d-1$ (see Proposition 3.5). By the definition of Γ' , the $d^2 \times d^2$ matrix representing ζz^j should be upper triangular modulo $1/y$. Now, $(\zeta z^j) z (\zeta z^j)^{-1} = \frac{\zeta}{\phi(\zeta)} z$, so $\zeta \in \mathbb{F}_q^\times$. Furthermore for every $a \in \mathbb{F}_{q^d}$, $(\zeta z^j) a (\zeta z^j)^{-1} = \phi^j(a)$, proving that $j = 0$ and ζz^j is central; thus $\Gamma' \cap K = 1$. This proves that Γ' acts simply transitively on $G(F)/K$, and since $\Gamma \subseteq \Gamma'$, these groups are equal. \square

Proposition 4.9. *The group Γ is a normal subgroup of $G'(R)$, and $G'(R) = G'(R_T) \rtimes \Gamma$. Moreover,*

$$(12) \quad G'(R_0) = G'(R).$$

Proof. Let $g \in G'(R)$, and write $g = \gamma a$ for $\gamma \in \Gamma \subseteq G'(R)$ and $a \in K$. Then $a = \gamma^{-1} g \in K \cap G'(R) = G'(R_T)$, by Equation (7), and $G'(R) = \Gamma \cdot G'(R_T)$.

The generators of Γ are permuted by $G'(R_T)$: $(\zeta_i z^j) b_u (\zeta_i z^j)^{-1} = b_{\zeta_i \phi^j(u)}$, so $\Gamma \triangleleft G'(R)$. Moreover, $\Gamma \cap G'(R_T) \subseteq \Gamma \cap K = 1$, proving the second claim.

Finally, $G'(R_0) \subseteq G'(R)$ but direct inspection using Equation (9), shows that Γ and $G'(R_T)$ are contained in $\text{GL}_{d^2}(R_0)$ (see [CS, Thm. 2.6]). \square

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\det} & F^\times / F^{\times d} \mathcal{O}^\times \\
\downarrow \gamma \mapsto \gamma[L_0] & \searrow \delta & \downarrow y \mapsto 1 \\
B_0 & \xrightarrow{\varrho} & \mathbb{Z}/d\mathbb{Z}
\end{array}$$

FIGURE 3. Determinant and coloring

Since Γ acts simply transitively, we can identify \mathcal{B}^0 with Γ by $\gamma \mapsto \gamma[L_0]$. Recall that by Equation (5), $\Gamma \subseteq \mathrm{PGL}_d(F)$. The determinant (modulo scalars) thus takes $\gamma \mapsto \det(\gamma) \cdot F^{\times d} \in F^\times / F^{\times d}$. Considering this modulo \mathcal{O}^\times (which is superfluous if d is prime to q), we obtain the cyclic group $F^\times / F^{\times d} \mathcal{O}^\times \cong \langle y \rangle / \langle y^d \rangle$. Moreover, since $\varrho(b_u x) = \varrho(x) + 1$ and $N(b_u) \equiv y \pmod{\mathcal{O}^\times}$ by Corollary 4.2, the diagram in Figure 3 commutes. We let δ denote the color map $\Gamma \rightarrow \mathbb{Z}/d\mathbb{Z}$. Letting $\Gamma_1 = \mathrm{Ker}(\delta)$, the short sequence

$$1 \longrightarrow \Gamma_1 \longrightarrow \Gamma \xrightarrow{\delta} \mathbb{Z}/d\mathbb{Z} \longrightarrow 1$$

is exact.

Corollary 4.10. $[\Gamma : \Gamma_1] = d$.

5. THE GEOMETRY OF Γ

By definition, Γ is generated by the elements $b_u = u(1 - z^{-1})u^{-1}$ of $G'(R)$, where u ranges over $\mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times$.

Since $b_u = 1 - \frac{u}{\phi(u)}z^{-1}$, for every u , there is a unique $r \in \mathbb{F}_{q^d}^\times$ with $N(r) = 1$, such that $b_u = b_{(r)} = 1 - rz^{-1}$, where $N = N_{\mathbb{F}_{q^d}/\mathbb{F}_q}$ is the norm map.

Remark 5.1. For every $u \neq v \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times$ there are unique $u', v' \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times$ such that

$$(13) \quad b_u b_{u'} = b_v b_{v'}.$$

Proof. The conceptual reason is that since $b_u L_0 \neq b_v L_0$, their intersection $L = b_u L_0 \cap b_v L_0$ is a submodule of index q^2 of L_0 (which contains yL_0), and so by Corollary 4.6 there are unique u', v' such that $L = b_u b_{u'} L_0 = b_v b_{v'} L_0$. Since Γ acts simply transitively, $b_u b_{u'} = b_v b_{v'}$.

Computationally, assume first $d > 2$. Since $(1 - rz^{-1})(1 - r'z^{-1}) = 1 - (r + r')z^{-1} + r\phi^{-1}(r')z^{-2}$, $b_{(r)}b_{(r')} = b_{(s)}b_{(s')}$ in Γ iff

$$r + r' = s + s' \quad \text{and} \quad r\phi^{-1}(r') = s\phi^{-1}(s').$$

The unique solution is $r' = \frac{s-r}{\phi s - \phi r} \phi(s)$ and $s' = \frac{s-r}{\phi s - \phi r} \phi(r)$, where r' and s' indeed have norm 1.

If $d = 2$ then (as elements of $\tilde{\Gamma}$) $b_{(-r)}b_{(r)} = 1 - \frac{r\phi(r)}{1+y}$, which is in the center $R = Z(\mathcal{A}(R))$. Therefore in Γ , $b_{(r)}^{-1} = b_{(-r)}$. Now the unique solution to the above equation is $b_{(r)}b_{(-r)} = 1 = b_{(s)}b_{(-s)}$. \square

The following is essentially a reformulation of the simply connectedness of \mathcal{B} .

Theorem 5.2. *Let Γ' be the abstract group generated by $\{x_u : u \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times\}$, with the relations $x_u x_u = x_{v'} x_{v'}$ whenever $b_u b_u = b_{v'} b_{v'}$ in Γ , and a single relation $x_{u_1} \dots x_{u_d} = 1$ for some u_1, \dots, u_d such that*

$$(14) \quad r = b_{u_1} \dots b_{u_d} = 1$$

in Γ . Then $x_u \mapsto b_u$ is an isomorphism $\Gamma' \rightarrow \Gamma$.

Proof. Since Γ acts simply transitively, $b_{u_1}^{\epsilon_1} \dots b_{u_n}^{\epsilon_n} = 1$ iff $b_{u_1}^{\epsilon_1} \dots b_{u_n}^{\epsilon_n} L = y^i L$ for some lattice L and $i \in \mathbb{Z}$.

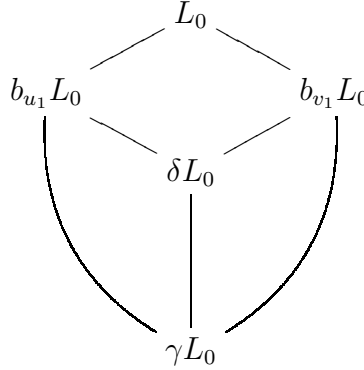


FIGURE 4. Proof of Theorem 5.2

We first claim that every equation $\gamma = b_{u_1} \dots b_{u_n} = b_{v_1} \dots b_{v_n}$ (both expressions of the same length) can be obtained from the relations of type (13). Indeed, for $n \leq 2$ the claim is obvious.

If $u_n = v_n$ then we are done by induction, so we assume $u_n \neq v_n$. By the above remark there are u' and v' such that $\delta = b_{u_n} b_{u'} = b_{v_n} b_{v'}$, and then $\delta L_0 = b_{u_1} L_0 \cap b_{v_1} L_0$ is of index q^2 in L_0 , and contains γL_0 ; see Figure 4. By Corollary 4.6 there are w_3, \dots, w_n such that $\gamma L_0 = \delta b_{w_3} \dots b_{w_n} L_0$. By induction the equations $b_{u_2} \dots b_{u_n} = b_{u'} b_{w_3} \dots b_{w_n}$ and $b_{v_2} \dots b_{v_n} = b_{v'} b_{w_3} \dots b_{w_n}$ follow from the relations (13), and so is the case with $b_{u_1} b_{u_2} \dots b_{u_n} = b_{u_1} b_{u'} b_{w_3} \dots b_{w_n} = b_{v_1} b_{v'} b_{w_3} \dots b_{w_n} =$

$b_{v_1} b_{v_2} \dots b_{v_n}$, which proves our claim. In particular, all the relations of the form (14) are equivalent modulo the relations (13).

Let $u \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times$. Using Corollary 4.6 we see that b_u^{-1} is equal to a product of $d - 1$ generators. Given a relation, we can thus assume it has the form $b_{u_1} \dots b_{u_n} = 1$. Then $b_{u_1} \dots b_{u_n} L_0 = y^i L_0$ where $n = di$, so $b_{u_1} \dots b_{u_n} = r^i$ is an equality of elements of equal length, which follows from the relations (13) by the above claim. \square

Remark 5.3. *If d is a power of $\text{char } F$, then equation (14) takes a particularly simple form: $b_1^d = (1 - z^{-1})^d = 1 - z^{-d} = \frac{y}{1+y} \in R^\times$, so $b_1^d = 1$ in Γ .*

From the theorem, together with the case $d = 2$ of Remark 5.1, we obtain

Corollary 5.4. *If $d = 2$, then the defining relations of Γ are $b_{(r)}^{-1} = b_{(-r)}$. In particular, if q is odd then Γ is free on $\frac{q+1}{2}$ generators, and if q is even then Γ is a free product of $q + 1$ cyclic groups of order 2.*

6. FINITE QUOTIENTS OF \mathcal{B}

In this section we specify quotients of Γ , whose Cayley graphs (with respect to the generators b_u) define Ramanujan complexes. These quotients are shown to be subgroups of PGL_d which contain PSL_d , over a pre-determined finite local ring.

Recall the definitions of $\tilde{G}', \tilde{G}'_1, G'$ and G'_1 from Section 3. Let S be an R_T -algebra, and $I \triangleleft S$ an ideal. The map $S \rightarrow S/I$ induces an epimorphism of algebras $\mathcal{A}(S) \rightarrow \mathcal{A}(S/I)$. It also induces a homomorphism $\tilde{G}'(S) \rightarrow \tilde{G}'(S/I)$ (which, in general, is not onto). The kernel of this map is called the congruence subgroup of $\tilde{G}'(S)$ with respect to I and denoted by $\tilde{G}'(S, I)$. In a similar manner we have a map $G'(S) \rightarrow G'(S/I)$ defined by $aS^\times \mapsto (a + I\mathcal{A}(S))(S/I)^\times$, and the kernel is denoted $G'(S, I)$. Note that $\tilde{G}(S, I)$ is mapped into $G(S, I)$, but this map is not necessarily onto. We set $\tilde{G}'_1(S, I) = \tilde{G}'_1(S) \cap \tilde{G}'(S, I)$, and likewise $G'_1(S, I) = G'(S, I) \cap G'_1(S)$. (Again, note that $G'_1(S, I) = \{aS^\times : a \in \tilde{G}'_1(S), a \in S^\times + I\mathcal{A}(S)\}$, while $\tilde{G}'_1(S, I) = \{a : a \in \tilde{G}'_1(S), a \in 1 + I\mathcal{A}(S)\}$, so the map

$$\tilde{G}'_1(S, I) \rightarrow G'_1(S, I)$$

induced by $\tilde{G}'(S) \rightarrow G'(S)$ may not be onto). See Figure 5.

Let $I \triangleleft R$. We define the congruence subgroup of Γ to be

$$(15) \quad \Gamma(I) = \Gamma \cap G'(R, I),$$

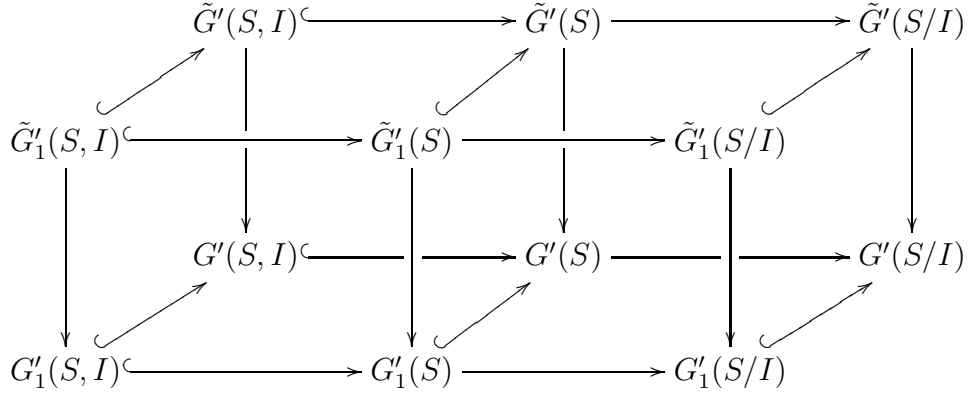


FIGURE 5. Group relations of Section 6

a normal subgroup of $G'(R)$ (by Proposition 4.9). Recall that by Proposition 4.8, Γ acts simply transitively on the vertices of $\mathcal{B} = G(F)/K$, where $K = G(\mathcal{O})$. The set of vertices in the quotient complex $\mathcal{B}_I = \Gamma(I) \backslash G(F)/K$ is isomorphic to $\Gamma/\Gamma(I)$. To make this an isomorphism of complexes, we need to define a complex structure on $\Gamma/\Gamma(I)$.

The generators $\{b_u\}$ correspond to the neighbors of color 1 of $[L_0]$. More generally, by Corollary 4.6 the neighbors of color k correspond to the products $b_{u_1} \dots b_{u_k}$ which are headers of a product of d generators which equals 1. The 1-skeleton of \mathcal{B}_I is thus the Cayley graph of $\Gamma/\Gamma(I)$ with respect to these generators (for $k = 1, \dots, d-1$), and there are ‘partial Laplacian operators’ A_k of $\Gamma/\Gamma(I)$, induced from the Hecke operators of \mathcal{B} . Frequently, these are the colored Laplacian operators (i.e. they can be defined for $\Gamma/\Gamma(I)$ directly); we address this issue towards the end of this section. The higher dimensional cells of $\Gamma/\Gamma(I)$ are then defined to make it a clique complex. Thus, $\Gamma(I) \backslash G(F)/K$ and $\Gamma/\Gamma(I)$ become isomorphic complexes.

Remark 6.1. *If $f(\lambda) \in \mathbb{F}_q[\lambda]$ is divisible by λ , then $G'(R_0, \langle f(1/y) \rangle) \subseteq G'(R_0, \langle 1/y \rangle) \subseteq \Gamma$, by Definition 4.7 of Γ' and Equation (12). In this case $\Gamma(I) = G'(R_0, I)$ (where $I = \langle f(1/y) \rangle$) so $\Gamma(I) \backslash \mathcal{B} \cong \Gamma/\Gamma(I)$.*

Let $C = \{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_i| = 1, z_1 \dots z_d = 1\}$ and $\sigma : C \rightarrow \mathbb{C}^{d-1}$ the map defined by $(z_1, \dots, z_d) \mapsto (\lambda_1, \dots, \lambda_{d-1})$, where

$$\lambda_k = q^{k(d-k)/2} \sigma_k(z_1, \dots, z_d)$$

and σ_k are the symmetric functions. Then $\mathfrak{S}_d = \sigma(C)$ is the simultaneous spectrum of the Hecke operators A_1, \dots, A_{d-1} acting on \mathcal{B} [LSV, Thm. 2.9].

Recall [LSV] that \mathcal{B}_I is called a Ramanujan complex if the eigenvalues of every non-trivial simultaneous eigenvector $v \in L^2(\mathcal{B}_I)$, $A_k v = \lambda_k v$, satisfy $(\lambda_1, \dots, \lambda_{d-1}) \in \mathfrak{S}_d$. It can be shown that the closure of the union of the sets of simultaneous eigenvalues over any family of quotients of \mathcal{B} (with unbounded injective radius) contains \mathfrak{S}_d .

The following theorem relies on one of the main results of [LSV], for which we assume the global Jacquet-Langlands correspondence for function fields. See the introduction for more details.

Theorem 6.2. *For every $d \geq 2$ and every $0 \neq I \triangleleft R$, the Cayley complex of $\Gamma/\Gamma(I)$ is a Ramanujan complex.*

Proof. Write $I = Rp$ for some polynomial $p \in \mathbb{F}_q[y]$ which is prime to y and $1+y$, then $I \cap R_0$ is generated by $y^{-e}p$ where $e = \deg(p)$. Let $I' = \langle y^{-e-1}p \rangle \triangleleft R_0$. Set $N = G'(R_0, I')$, then $N \subseteq \Gamma$ by Remark 6.1, and hence $N \subseteq \Gamma(I)$ (this property is not satisfied by the more ‘natural’ candidate $G'(R_0, \langle y^{-e}p \rangle)$).

In the notation of [LSV], $D = \mathcal{A}(k)$ is the division algebra used to define $G' = D^\times/Z^\times$, F is the completion of k with respect to $\nu_0 = \nu_y$, and the ramification places are $T = \{\nu_{1/y}, \nu_{1+y}\}$. The ring R_0 is

$$R_0 = \bigcap_{\nu \in \mathcal{V} - \{\nu_0\}} (k \cap \mathcal{O}_\nu) = \mathbb{F}_q[1/y],$$

where \mathcal{V} is the set of valuations of k . Since

$$R = \bigcap_{\nu \in \mathcal{V} - T \cup \{\nu_0\}} (k \cap \mathcal{O}_\nu),$$

every ideal of R restricts to an ideal of R_0 which is prime to T .

Note that I' is prime to ν_{1+y} (since $\nu_{1+y}(p) = 0$). In [LSV, Thm. 1.3(a)] we prove that $X = N \backslash \mathcal{B} = N \backslash \Gamma K / K$ is Ramanujan. This complex is isomorphic to Γ/N by $N\gamma K \mapsto \gamma N$, so Γ/N is Ramanujan. Since $N \subseteq \Gamma(I)$, $\Gamma/\Gamma(I)$ is also Ramanujan. \square

Our next goal is to identify $\Gamma/\Gamma(I)$ as an abstract group. We assume that I is a power of a prime ideal (so R/I is a finite local ring). Let $\langle p \rangle = \sqrt{I}$ denote the radical of I , so $I = \langle p^s \rangle$ for some $s \geq 1$, where $p \in \mathbb{F}_q[y]$ is prime. By definition of $\Gamma(I)$, we have that

$$\begin{aligned} \Gamma/\Gamma(I) &= \Gamma/(\Gamma \cap G'(R, I)) \\ &\cong (\Gamma \cdot G'(R, I))/G'(R, I) \\ &\subseteq G'(R)/G'(R, I) \\ &\hookrightarrow G'(R/I) \\ &\cong \mathrm{PGL}_d(R/I), \end{aligned}$$

where the final isomorphism follows from the definition of $G'(R/I)$ as $\mathcal{A}(R/I)^\times$ modulo center, and Wedderburn's theorem which implies that $\mathcal{A}(R/I) \cong M_d(R/I)$.

Theorem 6.3. *The map $G'_1(R) \rightarrow G'_1(R/I)$ is onto.*

Proof. Let \mathbb{A} denote the ring of adèles over k , and $\mathbb{A}_0 = \mathbb{A}/F$. Since \tilde{G}'_1 is connected, simply connected and simple, and $\tilde{G}'_1(F)$ is non-compact, the strong approximation theorem ([PR, Thm. 7.2], [Pr]) asserts that $\tilde{G}'_1(k)\tilde{G}'_1(F)$ is dense in $\tilde{G}'_1(\mathbb{A})$ (see [LSV, Sec. 3.2]). Therefore, $\tilde{G}'_1(k)$ is dense in $\tilde{G}'_1(\mathbb{A}_0)$. Let $U = \tilde{G}'_1(k_{1/y})\tilde{G}'_1(k_{1+y})\prod_{\nu \neq \nu_{1/y}, \nu_{1+y}, \nu_y} \tilde{G}'_1(\mathcal{O}_\nu)$, an open subgroup of $\tilde{G}'_1(\mathbb{A}_0)$. Then $\tilde{G}'_1(R) = \tilde{G}'_1(k) \cap U$ is dense in U .

In particular, $G'_1(R)$ is dense in $G'_1(\mathcal{O}_p)$, where \mathcal{O}_p is the ring of integers in the completion k_p . (Notice that $R/I \cong \mathcal{O}_p/P_p^s$ where P_p is the maximal ideal of \mathcal{O}_p). Since $G'_1(R/I) \cong \text{PSL}_d(R/I)$ which is finite, it is enough to show that the map from $G'_1(\mathcal{O}_p)$ to $G'_1(R/I) = G'_1(\mathcal{O}_p/P_p^s)$ is onto.

Let $c \in G'_1(\mathcal{O}_p/P_p^s)$, so c is an automorphism of $\mathcal{A}(\mathcal{O}_p/P_p^s)$ which is induced by an element \tilde{c} of norm one in $\mathcal{A}(\mathcal{O}_p/P_p^s)$. Lift it to an element \tilde{a} of $\mathcal{A}(\mathcal{O}_p)$, then $N(\tilde{a}) \in \mathcal{O}_p$ and is equivalent to 1 modulo P_p^s ; thus \tilde{a} is invertible and in $G'(\mathcal{O}_p)$.

If d is prime to $|\mathcal{O}_p/P_p|$, then there is some $\alpha \in \mathcal{O}_p$ such that $\alpha^d = N(\tilde{a})$, so $N(\alpha^{-1}\tilde{a}) = 1$. Thus the automorphism induced by $\alpha^{-1}\tilde{a}$ on $\mathcal{A}(\mathcal{O}_p)$ is in $G'_1(\mathcal{O}_p)$, call it a . Now a covers c , and we are done.

When d is not prime to $|\mathcal{O}_p/P_p|$, we construct an element of $\mathcal{A}(\mathcal{O}_p)$ by induction. Let $v \in \mathbb{F}_{q^d}$ be an element with $\text{tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(v) = 1$. Take $\tilde{a}_s = \tilde{a}$. Assume $N(\tilde{a}_i) \equiv 1 \pmod{p^i}$ for some $i \geq s$. Write $N(\tilde{a}_i) \equiv 1 + gp^i \pmod{p^{i+1}}$ for some $g \in R$. Let $\tilde{a}_{i+1} = (1 - vgp^i)\tilde{a}_i$, so $\tilde{a}_{i+1} \equiv \tilde{a}_i \pmod{p^i}$. Since $N(1 - vgp^i) = \prod_{j=0}^{d-1} (1 - \phi^j(vg)p^i) \equiv 1 - gp^i \pmod{p^{i+1}}$ by our choice of v , we have $N(\tilde{a}_{i+1}) \equiv (1 - gp^i)(1 + gp^i) \equiv 1 \pmod{p^{i+1}}$. Now let $\tilde{a}_0 = \lim \tilde{a}_i$. Then $N(\tilde{a}_0) = \lim N(\tilde{a}_i) = 1$, and $\tilde{a}_0 \equiv \tilde{a}_s \pmod{I}$, proving that the image of \tilde{a}_0 in $\tilde{G}'_1(\mathcal{O}_p/P_p^s)$ is \tilde{c} . Consequently, the image of \tilde{a}_0 in $G'_1(\mathcal{O}_p)$ covers c . \square

Let $H = G'(R, I)$, $H_1 = H \cap G'_1(R) = G'_1(R, I)$ and $\Gamma_1 = \Gamma \cap G'_1(R)$. (Note that the map $\tilde{G}'_1(R) \rightarrow G'(R)$ takes $\tilde{\Gamma}_1 = \tilde{\Gamma} \cap \tilde{G}'_1(R)$ to $\{\gamma R^\times : \gamma \in \tilde{\Gamma}, N(\gamma) = 1\}$ which may be a proper subgroup of $\Gamma_1 = \{\gamma R^\times : \gamma \in \tilde{\Gamma}, N(\gamma) \in R^{\times d}\}$).

In order to complete the identification of $\Gamma/\Gamma(I)$, we need to compute the index of some subgroups of $G'(R)$. From the definition of R , it follows that $R^\times = \mathbb{F}_q^\times \{y^i(1+y)^j : i, j \in \mathbb{Z}\}$. The norm map $\mathcal{A}(R)^\times \rightarrow R^\times$ is onto, since $N(b_u) = y/(1+y)$, $N(z) = 1+y$ and the

norm $\mathbb{F}_{q^d}^\times \rightarrow \mathbb{F}_q^\times$ is onto. Therefore, the map $G'(R) \rightarrow R^\times / R^{\times d}$, defined by $aR^\times \mapsto N(a)R^{\times d}$, is also onto. If aR^\times is in the kernel, then $N(a) = t^d$ for some $t \in R^\times$, and then $N(t^{-1}a) = 1$ and $aR^\times = t^{-1}aR^\times \in G'_1(R)$. It follows that $G'(R)/G'_1(R) \cong R^\times / R^{\times d}$, and

$$(16) \quad [G'(R):G'_1(R)] = [R^\times : R^{\times d}] = d^2(d, q-1).$$

On the other hand, from Corollary 4.10, Corollary 4.9 and Proposition 3.5, we obtain

$$(17) \quad [G'(R):\Gamma_1] = \frac{q^d - 1}{q - 1} d^2.$$

Corollary 6.4. Γ_1 is not contained in any congruence subgroup $G'_1(R, I)$ for $0 \neq I \triangleleft R$ ($I \neq R$).

Proof. We may assume I is maximal. By Theorem 6.3 the index of $G'_1(R, I)$ in $G'_1(R)$ is $|\mathrm{PSL}_d(R/I)| \geq |\mathrm{PSL}_d(\mathbb{F}_q)| > \frac{q^d - 1}{(q-1)(d, q-1)}$, so that $[G'(R):G'_1(R, I)] = d^2(d, q-1)[G'_1(R):G'_1(R, I)] > \frac{q^d - 1}{q-1} d^2 = [G'(R):\Gamma_1]$. \square

Corollary 6.5. For every $I \triangleleft R$, $G'_1(R) = \Gamma_1 \cdot H_1$ where $H_1 = G'_1(R, I)$.

Proof. By Proposition 4.9, Γ and H are normal in $G'(R)$, so $\Gamma_1 H_1$ is normal in $G'_1(R)$, and $\Gamma_1 H_1 / H_1$ is normal in $G'_1(R) / H_1 \cong G'_1(R/I)$, which is isomorphic to $\mathrm{PSL}_d(R/I)$.

Recall that $I = \langle p \rangle^s$ where p is prime, and $R/\langle p \rangle = \mathbb{F}_{q^\alpha}$ for an appropriate α . First assume that $\mathrm{PSL}_d(\mathbb{F}_{q^\alpha})$ is simple, so that $d > 2$ or $q^\alpha > 3$. The group $\mathrm{PSL}_d(R/I)$ has normal subgroups $N_j = \mathrm{Ker}(\mathrm{PSL}_d(R/I) \rightarrow \mathrm{PSL}_d(R/\langle p \rangle^j))$. Now, the composition factors are $\mathrm{PSL}_d(R/I)/N_1 \cong \mathrm{PSL}_d(R/\langle p \rangle)$, and the N_j/N_{j+1} for $j = 1, \dots, s-1$, which are all isomorphic to the zero-trace subalgebra of $M_d(\mathbb{F}_{q^\alpha})$, and are irreducible $\mathrm{PSL}_d(\mathbb{F}_{q^\alpha})$ -modules. If $N \trianglelefteq \mathrm{PSL}_d(R/I)$ is a normal subgroup which is not contained in N_1 , then $\mathrm{PSL}_d(R/I)/N_1 \cong N/(N \cap N_1)$ is a group of order power of q , which thus has a normal subgroup of index q , producing an impossible composition factor.

This proves $\Gamma_1 H_1 = G_1$, for otherwise $\Gamma_1 H_1 / H_1$ is a proper normal subgroup of G_1 / H_1 , which is therefore contained in $N_1 / H_1 = G'_1(R, \sqrt{I})$, contradicting the last corollary.

In the special case $d = 2$ and $|R/I| = 3$, $G'_1(R)/H_1 \cong A_4$ and has no non-trivial normal subgroups of index $\leq 2 = [G'_1(R):\Gamma_1]$. The case $|R/I| = 2$ is impossible since the prime generator p must be different than y and $1 + y$, and so it cannot be linear over \mathbb{F}_2 . \square

There is a delicate point here: the claim that $G'_1(R) = \Gamma_1 \cdot G'_1(R, I)$ holds for every proper ideal of R , but cannot be extended to the wider class of ideals of R_0 (in the form $G'_1(R) = \Gamma_1 \cdot G'_1(R_0, I)$); indeed, we saw in the proof of Theorem 6.2 that $G'_1(R_0, I) \subseteq \Gamma_1$ for suitable ideals of R_0 .

From this and the definition of Γ_1 and H_1 , it follows that for the subgroups of $G'(R)$ appearing in Figure 6, the intersections and products can be read from the lattice. The only non-trivial equality is $\Gamma_1 H_1 (\Gamma \cap H) = \Gamma_1 H \cap \Gamma H_1$, and this too is easily checked (if $\gamma_1 h = \gamma h_1$ for $\gamma_1 \in \Gamma_1$, $\gamma \in \Gamma$, $h \in H$ and $h_1 \in H_1$, then $\gamma^{-1} \gamma_1 = h_1 h^{-1} \in \Gamma \cap H$ so $\gamma_1 (h_1 h^{-1})^{-1} h_1 \in \Gamma_1 H_1 (\Gamma \cap H)$).

In particular, the quotients of the extensions denoted by double lines are isomorphic to $G_1/H_1 = G'_1(R)/G'_1(R, I) \cong G'_1(R/I) \cong \text{PSL}_d(R/I)$.

Let $L = R/I$. We now observe that

$$\text{PSL}_d(L) \cong \Gamma_1 H_1 / H_1 \cong \Gamma_1 H / H \subseteq \Gamma H / H \subseteq G'(R) / H \subseteq \text{PGL}_d(L),$$

so we proved:

Theorem 6.6. $\Gamma_I = \Gamma / \Gamma(I) \cong \Gamma H / H$ is a subgroup of $\text{PGL}_d(L)$ which contains $\text{PSL}_d(L)$.

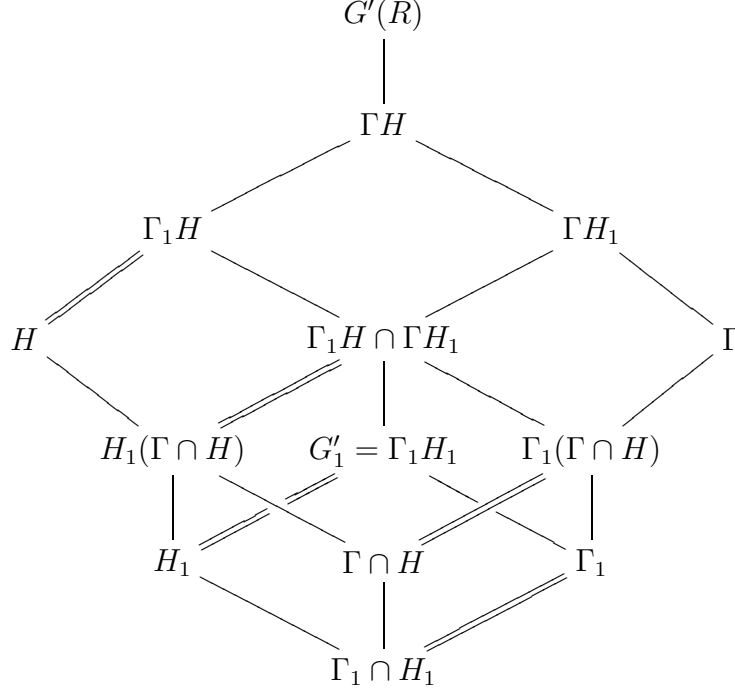
Proposition 6.7. Let r denote the order of $y/(1+y)$ in $L^\times / L^{\times d}$. Then $[\Gamma_I : \text{PSL}_d(L)] = r$.

Proof. The composition $\Gamma_I \cong \Gamma H / H \hookrightarrow \text{PGL}_d(R/I) \xrightarrow{N} L^\times / L^{\times d}$ carries Γ_I to the subgroup generated by $N(b) = y/(1+y)$, and its kernel is $\Gamma_1 H / H \cong \text{PSL}_d(L)$. \square

The index of $\text{PSL}_d(L)$ in Γ_I can be interpreted in another way. As mentioned above, the color map $\delta : \Gamma \rightarrow \mathbb{Z}/d$ of Figure 3 is in fact an isomorphism $\Gamma/\Gamma_1 \rightarrow \mathbb{Z}/d$, so it induces an isomorphism $\Gamma H / \Gamma_1 H \cong \Gamma / \Gamma_1 (\Gamma \cap H) \rightarrow \mathbb{Z}/r$. In other words, there is an r -coloring of $\Gamma H / H$.

A remark about colors is in order. For the building \mathcal{B} , there is a $d \times d$ matrix n_{ij} of integers, such that every vertex of color i has n_{ij} neighbors of color j (in fact $n_{ij} = \lfloor \binom{d}{i-j} \rfloor_q - \delta_{ij}$, where $\lfloor \binom{d}{k} \rfloor_q$ is the number of subspaces of dimension k of \mathbb{F}_q^d), and in particular $n_{ii} = 0$ (so the graph is d -partite).

A quotient complex \mathcal{B}_I which has a d_1 -coloring still satisfies this property, for the ‘folded’ $d_1 \times d_1$ matrix whose (i', j') entry is the sum of n_{ij} over $i \equiv i' \pmod{d_1}$ and $j \equiv j' \pmod{d_1}$. Naturally the most interesting case is when the quotient has d colors, which the above theorem guarantees quite often:

FIGURE 6. Subgroups of $G'(R)$

Corollary 6.8. *If $y/(1+y)$ has order d in $L^\times/L^{\times d}$, then there is a well defined color epimorphism $\delta_L : \Gamma/\Gamma(I) \rightarrow \mathbb{Z}/d\mathbb{Z}$.*

7. CONSTRUCTION WITH GIVEN INDEX OVER $\mathrm{PSL}_d(L)$

We combine the results obtained so far to prove Theorem 1.1, in the following, stronger version (the assumption $q^e > 4d^2 + 1$ is only needed here to control the index r).

Theorem 7.1. *Let q be a prime power, $d \geq 2$, $s \geq 1$, $e \geq 1$ ($e > 1$ if $q = 2$), and r a divisor of $(d, q^e - 1)$. Assume d is prime to q , or $s = 1$. Also assume that $q^e > 4d^2 + 1$.*

Then, for a suitable prime polynomial $p \in \mathbb{F}_q[y]$ of degree e and for $L = \mathbb{F}_q[y]/\langle p(y)^s \rangle$, the (unique) subgroup G of $\mathrm{PGL}_d(L)$ which has index r over $\mathrm{PSL}_d(L)$ has a set of $\begin{bmatrix} d \\ 1 \end{bmatrix}_q + \begin{bmatrix} d \\ 2 \end{bmatrix}_q + \cdots + \begin{bmatrix} d \\ d-1 \end{bmatrix}_q$ generators, such that the Cayley graph of G with respect to these generators determines a Ramanujan complex.

Let $J \triangleleft R$ be a prime ideal and $I = J^s$ for some $s \geq 1$. Let $L_0 = R/J \cong \mathbb{F}_{q^e}$, the residue field of $L = R/I$.

Notice that $\mathrm{PGL}_d(L)/\mathrm{PSL}_d(L) \cong L^\times/L^{\times d}$ is usually not a cyclic group, so Γ_I may not be the full group $\mathrm{PGL}_d(L)$. However, in two important cases, namely if d is prime to q or if $L = L_0$, the quotient $L^\times/L^{\times d} \cong L_0^\times/L_0^{\times d}$ is cyclic.

We want to choose the generator of J so that $\Gamma/\Gamma(I)$ has a pre-determined index over $\mathrm{PSL}_d(L)$. Write $J = \langle p \rangle$, where p is a prime polynomial different than y and $1 + y$, since $R = \mathbb{F}_q[y, 1/y, 1/(1+y)]$. In particular, if $q = 2$ then we must assume $e \geq 2$.

It is convenient to have a uniform bound on the number of non-generators of a finite field.

Lemma 7.2. *Let q be a prime power and $e > 1$, and let \mathbb{F}_{q^e} be a field of order q^e . There are less than $2q^{e/2}$ elements of \mathbb{F}_{q^e} which do not generate it over \mathbb{F}_q .*

Proof. We need to bound the size of the union of the maximal subfields of dimensions e/p , for the prime divisors p of e . If e is a prime or equals 4, then the statement is obvious, so we assume $e \geq 6$ and is not a prime. Let $C = 9/q$.

We claim that for every prime $p \geq 3$ dividing e , $q^{e/p} < Cp^{-2}q^{e/2}$. Indeed, since e is not a prime, $p \leq e/2$. Now, in the range $3 \leq p \leq e/2$, $h(p) = 2\log(p) - (e/2 - e/p)\log(q)$ obtains its maximum on one of the boundaries, and so is bounded by

$$\max(h(3), h(e/2)) = \max(\log(9) - e\log(q)/6, 2\log(e/2) - (e/2 - 2)\log(q)).$$

This decreases with e , and is thus bounded by its value at $e = 6$, namely $\log(9/q)$.

Since $\sum_{p \geq 3} p^{-2} < 2/9$ (summing over all the primes), we have that $\sum_{p|e} q^{e/p} < (1 + C \sum p^{-2})q^{e/2} < (1 + 2/q)q^{e/2}$. \square

Proposition 7.3. *Let q, d, s, r and e be as in Theorem 7.1.*

If $q^e > 4d^2 + 1$, then there is a polynomial $p \in \mathbb{F}_q[y]$ of degree e , such that for $J = \langle p \rangle \triangleleft R$ and $I = J^s$, $L_0 = R/J$ is the field of order q^e , and $[\Gamma_I : \mathrm{PSL}_d(L)] = r$ where $L = R/I$.

Proof. The assumption that $s = 1$ or d is prime to q guarantees that $L^\times/L^{\times d} \cong L_0^\times/L_0^{\times d}$.

Let α be an element of \mathbb{F}_{q^e} whose order in $\mathbb{F}_{q^e}^\times/\mathbb{F}_{q^e}^{\times d}$ is r . Let $\alpha_1 = \frac{\alpha}{1-\alpha}$. The map $\alpha \mapsto \alpha_1$ is one-to-one.

There are $\frac{q^e-1}{(d, q^e-1)}\phi(r) \geq \frac{q^e-1}{(d, q^e-1)}$ suitable elements α , and α_1 belongs to a proper subfield of \mathbb{F}_{q^e} in less than $2q^{e/2} \leq \frac{q^e-1}{(d, q^e-1)}$ cases. We can thus assume α_1 generates \mathbb{F}_{q^e} as a field. Then, the minimal polynomial $f \in \mathbb{F}_q[y]$ of α_1 has degree e . Take $I = \langle f^s \rangle$. Then $L_0 = R/\langle f \rangle$

is isomorphic to \mathbb{F}_{q^e} , and the image of y in L_0 satisfies $y/(1+y) \mapsto \alpha_1/(1+\alpha_1) = \alpha$. It has order r in $L_0^\times/L_0^{\times d}$, and we are done by Proposition 6.7. \square

Recall that the group $\Gamma \subseteq G'(R)$ (Definition 4.4) is generated by the $\begin{bmatrix} d \\ 1 \end{bmatrix}_q = \frac{q^d-1}{q-1}$ elements $b_u = u(1-z^{-1})u^{-1}$, which can be viewed as $d^2 \times d^2$ matrices over $R_0 = \mathbb{F}_q[1/y]$; see Proposition 3.3. The quotient $G'_1(R)/G'_1(R, I)$ was computed in Section 6. For an explicit identification of this group, it is worth mentioning that $G'(R)/G'(R, I) \cong i_R G'(R)/i_R G'(R, I)$ (i_R was defined in Section 3), and likewise for G'_1 . Therefore when dealing with this quotient we may view (the entries of) the generators of Γ as if they were elements of R/I .

Let G be the group generated by these matrices over $R_0/(y^{-e}p)^s$. Then $G \cong \Gamma_I$ and, therefore, has index r over $\mathrm{PSL}_d(L)$. For each $k = 1, \dots, d-1$, the generators of color k of G are the products $b_{u_1} \dots b_{u_k}$, which can be extended to a product of d generators which equals 1 (see Section 5 and Proposition 2.3); in particular, every b_u is a generator of color 1. See [CS, Sec. 4] for an alternative description of the generators (as elements of $\mathcal{A}(k)$).

The description of the complex constructed from the Cayley graph is given at the beginning of Section 6. Theorem 6.2 then completes the proof of Theorem 7.1.

8. SPLITTING $\mathcal{A}(R)$

So far Γ is viewed as a group of $d^2 \times d^2$ matrices, and the representation of Γ as $d \times d$ matrices is over the completion F of k . To facilitate the explicit constructions of finite quotients in the next section, we now present a splitting extension of dimension d for $G'(k)$ (and $G'(R)$).

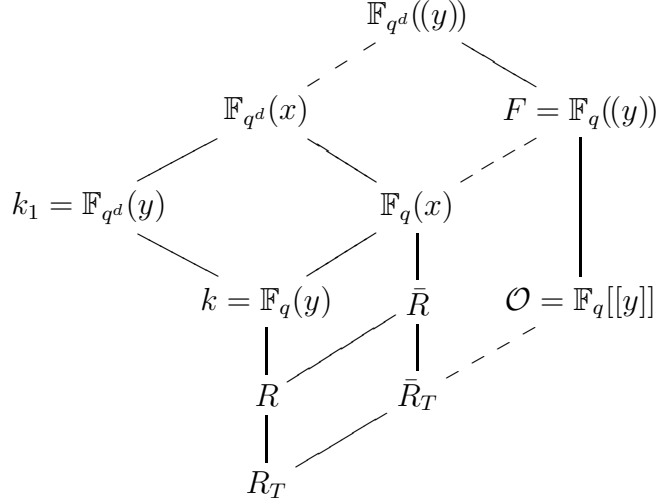
Fix an element $\beta \in \mathbb{F}_{q^d}$ such that $\mathrm{tr}(\beta) \in \mathbb{F}_q$ is non-zero. Let $\bar{k}_1 = \mathbb{F}_{q^d}(x)$, with the obvious action of $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$, so the fixed field is $\bar{k} = \mathbb{F}_q(x)$. Then the norm $N_{\mathbb{F}_{q^d}(x)/\mathbb{F}_q(x)}(1 + \beta x)$ is a polynomial of degree d in $\mathbb{F}_q[x]$, which has the form $1 + t_1 x + \dots + t_d x^d$ for some $t_i \in \mathbb{F}_q$. Then $t_1 = \mathrm{tr}(\beta) \neq 0$ by assumption, and $t_d = N_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\beta) \neq 0$, since $\beta \neq 0$.

We embed $k = \mathbb{F}_q(y)$ into \bar{k} by mapping

$$(18) \quad y \mapsto t_1 x + \dots + t_d x^d.$$

Now k becomes a subfield of codimension d of $\mathbb{F}_q(x)$.

Proposition 8.1. *Taking the completion, $\mathbb{F}_q((x)) = \mathbb{F}_q((y))$, and even $\mathbb{F}_q[[x]] = \mathbb{F}_q[[y]]$.*

FIGURE 7. Subrings of $\mathbb{F}_{q^d}((y))$

Proof. Before giving the proof, we should explain the notation. By $\mathbb{F}_q((x))$ we mean, as usual, the field of Taylor series $\{\sum_{i=-v}^{\infty} \alpha_i x^i\}$ with coefficients from \mathbb{F}_q ; $\mathbb{F}_q[[x]]$ is the subring of elements of the form $\sum_{i=0}^{\infty} \alpha_i x^i$. As abstract fields, it trivially holds that $\mathbb{F}_q((y)) \cong \mathbb{F}_q((x))$. But now $\mathbb{F}_q((y))$ is the subfield of $\mathbb{F}_q((x))$ consisting of Taylor series in the element y of Equation (18).

Working in $\mathbb{F}_q((y))$, Equation (18) has a solution for x by Hensel's lemma (since the derivation at zero, namely t_1 , is non-zero). In fact, the solution has the form $x = \frac{1}{t_1}y - \frac{t_2}{t_1^2}y^2 + \dots$, so that $x \in \mathbb{F}_q[[y]]$. \square

Tensoring with \mathbb{F}_{q^d} , we also have that $\mathbb{F}_{q^d}((y)) = \mathbb{F}_{q^d}((x))$. Define $\bar{R}_T = \mathbb{F}_q[x, 1/(1+y)]$ and $\bar{R} = \mathbb{F}_q[x, 1/y, 1/(1+y)]$, so that $R_T \subseteq \bar{R}_T$ and $R \subseteq \bar{R}$. Since $1+y$ is invertible in $\mathbb{F}_q[[y]]$, $\bar{R}_T \subseteq \mathbb{F}_q[[y]]$. The rings mentioned so far appear in Figure 7.

Notice that $\mathbb{F}_q(x)$ is a splitting field of $\mathcal{A}(k)$, as $1+y = N(1+\beta x)$ is a norm in the extension $\mathbb{F}_{q^d}(x)/\mathbb{F}_q(x)$. We thus obtain the chain of inclusions

$$(19) \quad \begin{aligned} \mathcal{A}(R) &\subseteq \mathcal{A}(k) \hookrightarrow \mathcal{A}(k) \otimes_k \mathbb{F}_q(x) \\ &\cong M_d(\mathbb{F}_q(x)) \hookrightarrow M_d(\mathbb{F}_q((x))) = M_d(\mathbb{F}_q((y))), \end{aligned}$$

refining Equation (5). The last equality in this chain follows from Proposition 8.1.

$$\begin{array}{ccc}
\mathcal{A}(\bar{k}) & \xrightarrow{\cong} & \text{End}_{\bar{k}}(\bar{k} \otimes \mathbb{F}_{q^d}) \\
\uparrow & & \uparrow \\
\mathcal{A}(\bar{R}_T) & \longrightarrow & \text{End}_{\bar{R}_T}(\bar{R}_T \otimes \mathbb{F}_{q^d})
\end{array}$$

FIGURE 8. Maps in Proposition 8.2

In fact, $\mathcal{A}(\bar{R}_T)$ is already split, and we now present an explicit isomorphism $\mathcal{A}(\bar{R}_T) \cong M_d(\bar{R}_T)$.

Proposition 8.2. *Let $M = \bar{R}_T \otimes \mathbb{F}_{q^d}$, and let $\rho: M \rightarrow \text{End}_{\bar{R}_T}(M)$ be the regular representation, defined by $\rho: r \otimes u \mapsto \rho_r \otimes \rho_u$, where $\rho_r(r') = rr'$ for $r' \in \bar{R}_T$, and $\rho_u(u') = uu'$ for $u \in \mathbb{F}_{q^d}$.*

The map ρ is extended by $z \mapsto 1 \otimes \phi + \rho_x \otimes \rho_\beta \phi$ to an isomorphism

$$\mathcal{A}(\bar{R}_T) \xrightarrow{\sim} \text{End}_{\bar{R}_T}(M).$$

Proof. By definition $\rho(z) = \rho(1 + \beta x) \cdot (1 \otimes \phi)$, so that $\rho(z)^{-1} = (1 \otimes \phi^{-1}) \cdot \rho(1 + \beta x)^{-1}$. We need to check the defining relations of $\mathcal{A}(\bar{R}_T)$. Let $r \otimes u, r' \otimes u' \in M$; then

$$\begin{aligned}
\rho(z)\rho(r \otimes u)(r' \otimes u') &= \rho(z)(rr' \otimes uu') \\
&= (1 + \beta x)(rr' \otimes \phi(uu')) \\
&= (1 + \beta x)(r \otimes \phi(u))(r' \otimes \phi(u')) \\
&= \rho(r \otimes \phi(u))(1 + \beta x)(r' \otimes \phi(u')) \\
&= \rho(r \otimes \phi(u))\rho(z)(r' \otimes u'),
\end{aligned}$$

so that $\rho(z)\rho(r \otimes u)\rho(z)^{-1} = \rho(r \otimes \phi(u))$. Also, $\rho(z)^d = (\rho(1 + \beta x) \cdot (1 \otimes \phi))^d = \rho(N(1 + \beta x)) \cdot (1 \otimes \phi)^d = \rho(1 + y)$.

To check that the extended ρ is an embedding, let $\bar{k} = \mathbb{F}_q(x)$, and recall that $\mathcal{A}(\bar{k}) \cong \text{End}_{\bar{k}}(\bar{k} \otimes \mathbb{F}_{q^d})$ by the norm condition. The result then follows from the commutativity of the diagram in Figure 8, and the fact that $\mathcal{A}(\bar{R}_T)$ and $\text{End}_{\bar{R}_T}(M)$ have the same rank d^2 over \bar{R}_T . \square

Since $\bar{R}_T \otimes \mathbb{F}_{q^d}$ is a free module of rank d over \bar{R}_T , we obtain an isomorphism $\mathcal{A}(\bar{R}_T) \cong M_d(\bar{R}_T)$.

Example 8.3. *Suppose $q = 7$ and $d = 3$, and let $\alpha = \sqrt[3]{2}$ over \mathbb{F}_7 , so that $\mathbb{F}_{7^3} = \mathbb{F}_7[\alpha]$. If ϕ is the Frobenius automorphism, we have that $\phi(\alpha) = \alpha^7 = 4\alpha$. As a normal basis of $\mathbb{F}_{7^3}/\mathbb{F}_7$ we take $\zeta_0 = 1 + \alpha + \alpha^2$ and $\zeta_i = \phi^i(\zeta_0)$. We choose $\beta = \alpha - 2$ (so that $\text{tr}(\beta) = 1$). Of course*

$\{\zeta_i\}$ form a basis of $\bar{R}_T \otimes \mathbb{F}_{7^3}$ as a free module over \bar{R}_T . Computing directly, we find that $N(1+\beta x) = 1+x-2x^2+x^3$, so that $y = x-2x^2+x^3$.

In order to present an embedding of $\mathcal{A}(\bar{R}_T) = \mathbb{F}_{7^3}[x, 1/(1+y)][z]$ into $M_3(\mathbb{F}_7[x, 1/(1+y)])$, we need to describe the images of \mathbb{F}_{7^3} , x , $1/(1+y)$ and z . For the subfield, check that

$$\alpha \mapsto \begin{pmatrix} 6 & 3 & 6 \\ 5 & 5 & 6 \\ 5 & 3 & 3 \end{pmatrix}$$

defines the embedding $\mathbb{F}_{7^3} \rightarrow M_3(\mathbb{F}_7)$ via the basis we chose. Secondly, x and $1/(1+y)$ are scalars in \bar{R}_T and so are mapped to scalar matrices. Finally, since $\phi(\zeta_i) = \zeta_{i+1}$, by the above proposition we have

$$\begin{aligned} z &\mapsto \rho(1+\beta x) \cdot (1 \otimes \phi) \\ &= \begin{pmatrix} 1+4x & 3x & 6x \\ 5x & 1+3x & 6x \\ 5x & 3x & 1+x \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3x & 6x & 1+4x \\ 1+3x & 6x & 5x \\ 3x & 1+x & 5x \end{pmatrix}. \end{aligned}$$

One can check that over \bar{R}_T , z^3 is the scalar matrix $1+y$.

9. EXPLICIT CONSTRUCTIONS

In this section we show how to choose the generator $p^s \in \mathbb{F}_q[y]$ of I , such that $\Gamma_I = \Gamma/\Gamma(I)$ will have an explicit representation as $d \times d$ matrices over L . The construction works for arbitrary q , d , $e = \deg(p)$ and $s \geq 1$, but we do not try to control the index r of Γ_I over $\text{PSL}_d(L)$ (which can be computed by Proposition 6.7).

Recall that $H = G'(R, I)$. Our current realization of $\Gamma H/H$ as a subgroup of $\text{PGL}_d(R/I)$ is inexplicit, as we use the abstract isomorphism $G'(R/I) \cong \text{PGL}_d(R/I)$. We do have an explicit embedding of i_R of $G'(R)$ into $d^2 \times d^2$ matrices over R , as described before Proposition 3.3, but for practical reasons it is more convenient to have a representation of dimension d .

In Proposition 8.2 (and Example 8.3) we gave an explicit isomorphism of $\mathcal{A}(\bar{R}_T)$ to matrices over \bar{R}_T , which in particular embeds $G'(R) \hookrightarrow \text{PGL}_d(\bar{R})$. Given $I \triangleleft R$, there is an ideal $\bar{I} \triangleleft \bar{R}$ such that $I = \bar{I} \cap R$ (since $\bar{R} = R[x]$ is an integral extension over R and \bar{R} is a principal ideal domain). Let $\bar{L} = \bar{R}/\bar{I}$, then there is a natural embedding $L \subseteq \bar{L}$, and we thus have an embedding $G'(L) \hookrightarrow \text{PGL}_d(\bar{L})$.

We will show that it is possible to choose I so that $\bar{L} \cong L$, where the image of x in L (which is what we need for the splitting map of Proposition 8.2) is explicitly identified.

We need some details on the possible choices of \bar{I} . Write $y(\lambda) = t_1\lambda + \dots + t_d\lambda^d$ for the polynomial defined by Equation (18), so $y(x)$ is our usual element y of $\mathbb{F}_q[x]$. If $p(\lambda) \in \mathbb{F}_q[\lambda]$ is irreducible and not equal to λ or $\lambda + 1$, then $p(y) \in R$ generates a prime ideal $J \triangleleft R$. Since the embedding of R into \bar{R} is by sending y to $y(x)$, J lifts to the ideal $\bar{J} = \langle p(y(x)) \rangle$ of \bar{R} . Let $p(y(\lambda)) = g_1(\lambda) \dots g_t(\lambda)$ be the decomposition into prime factors over \mathbb{F}_q and $\bar{J}_i = \langle g_i(x) \rangle \triangleleft \bar{R}$, then $\langle p(y(x)) \rangle = \bar{J}_1 \dots \bar{J}_t$, where each \bar{J}_i is a prime ideal of \bar{R} covering J . If α is a root of g_i in some extension, then $y(\alpha) \in \mathbb{F}_q[\alpha]$ is a root of $p(\lambda)$. Therefore p splits in $\mathbb{F}_q[\alpha]$ and thus $\deg(p(\lambda)) \mid [\mathbb{F}_q[\alpha] : \mathbb{F}_q] = \deg(g_i)$. This shows once more that $\bar{L}_0^i = \bar{R}/\bar{J}_i$ is an extension of $L_0 = R/J$, and in fact $[\bar{L}_0^i : L_0] = \deg(g_i)/\deg(p)$. We remark that the degrees of the g_i are not necessarily equal.

Let γ_j be the roots of $p(\lambda)$ in L_0 , then $p(\lambda) = \prod(\lambda - \gamma_j)$ and $p(y(\lambda)) = \prod(y(\lambda) - \gamma_j)$. In particular the roots of $p(y(\lambda))$ are roots of the factors $y(\lambda) - \gamma_j$. Let α be a root of $p(y(\lambda))$ coming from an irreducible factor $g_i(\lambda)$. Then $\bar{L}_0^i = \mathbb{F}_q[\alpha] = L_0[\alpha]$ (since a generator γ_j of L_0 equals $y(\alpha)$). It follows that $p(y(\lambda))$ has a factor g_i of degree $\deg(p)$ iff $\bar{L}_0^i = L_0$, iff $y(\lambda) - \gamma_j$ has a root in L_0 for some j (a condition which is easily seen to be independent of j).

We now show that $J = \langle p(y) \rangle \triangleleft R$ can be chosen so that $\bar{L}_0 = L_0$ and $\bar{L} = L$, which provides an explicit realization of Theorem 6.6.

Theorem 9.1. *Let q be a prime power and d an integer. Assume $q^e \geq 4d^2$. Then there are irreducible polynomials $p, g \in \mathbb{F}_q[\lambda]$ of degree e such that for every $s \geq 1$, the embedding $R \rightarrow \bar{R}$ induces an isomorphism $R/I \cong \bar{R}/\bar{I}$ for $I = \langle p(y)^s \rangle \triangleleft R$ and $\bar{I} = \langle g(x)^s \rangle \triangleleft \bar{R}$.*

Proof. Let \mathbb{F}_{q^e} be the field of order q^e . For every $\alpha \in \mathbb{F}_{q^e}$, consider the element $y(\alpha)$. Since $\deg(y(\lambda)) = d$, there are at most d elements α with the same $y(\alpha)$, so in particular (by Lemma 7.2), $y(\alpha)$ is not a generator of \mathbb{F}_{q^e} for less than $2dq^{e/2}$ choices of α . By the assumption on e , we can choose α for which $\gamma = y(\alpha)$ generates the field of q^e elements (so necessarily α is also a generator). Let $p(\lambda)$ and $g(\lambda)$ be the minimal polynomials of γ and α over \mathbb{F}_q , respectively. Since α and γ are generators, $\deg(p) = \deg(g) = e$.

Now let $I = \langle p(y)^s \rangle \triangleleft R$ and $\bar{I} = \langle g(x)^s \rangle \triangleleft \bar{R}$. Since $g(x) \mid p(y(x))$ and $g(x)$ is prime, $I = \bar{I} \cap R$, and so $L = R/I$ embeds into $\bar{L} = \bar{R}/\bar{I}$. Since $\deg(p) = \deg(g)$, $|L| = |\bar{L}|$, and so they are isomorphic. \square

The same choice of p and g works for every $s \geq 1$, so the proposition shows that in fact the fraction field $\bar{k} = k[x]$ of \bar{R} is contained in the completion k_p .

We conclude this section with a detailed algorithm giving the generators for a Cayley complex of $\mathrm{PGL}_d(L)$. Let q be a given prime power, $d \geq 2$, and $e \geq 1$. Assume $q^e \geq 4d^2$.

- Algorithm 9.2.** (1) In practice, the elements of \mathbb{F}_{q^d} are polynomials of degree at most d over \mathbb{F}_q , modulo a fixed irreducible polynomial of degree d . Fix a basis $\zeta_0, \dots, \zeta_{d-1}$ for \mathbb{F}_{q^d} over \mathbb{F}_q . Let ϕ be any generator of $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ (namely exponentiation by q^ℓ where ℓ is any fixed integer prime to d). Let $\beta \in \mathbb{F}_{q^d}$ be an element with $\beta + \dots + \phi^{d-1}(\beta) \neq 0$, let $t_1, \dots, t_d \in \mathbb{F}_q$ be defined by $(1 + \beta\lambda) \dots (1 + \phi^{d-1}(\beta)\lambda) = 1 + t_1\lambda + \dots + t_d\lambda^d$, and set $y(\lambda) = t_1\lambda + \dots + t_d\lambda^d$. For an element $c \in \mathbb{F}_{q^d}^\times$, $\rho(c) \in \mathrm{GL}_d(\mathbb{F}_q)$ is the matrix defined by $c \cdot \zeta_j = \sum_{i=0}^{d-1} (\rho(c))_{ij} \zeta_i$. Likewise, let $\varphi \in \mathrm{GL}_d(\mathbb{F}_q)$ be the matrix defined by $\phi(\zeta_j) = \sum_{i=0}^{d-1} \varphi_{ij} \zeta_i$.
- (2) Let $\alpha \in \mathbb{F}_{q^e}$ be an element such that $\gamma = y(\alpha)$ generates \mathbb{F}_{q^e} as a field (the existence of which is guaranteed by Theorem 9.1). A quick way to check this property is to verify that $\gamma^{q^{e'}} \neq \gamma$ for every proper divisor e' of e . Let $p(\lambda)$ and $g(\lambda)$ be the minimal polynomials (of degree e) of γ and α , respectively, over \mathbb{F}_q (found by Gaussian elimination on $1, \gamma, \dots, \gamma^e$ or $1, \alpha, \dots, \alpha^e$).
- (3) Fix $s \geq 1$. Let \bar{L} denote the ring of polynomials of degree $< es$ in the variable x , with operations modulo $g(x)^s$. Let $\rho(z) \in \mathrm{GL}_d(L)$ be the matrix $(1 + \rho(\beta)x) \cdot \varphi$, as in Proposition 8.2 and the example following it. Let $b = 1 - \rho(z)^{-1}$ (Gauss elimination can be used to invert $\rho(z)$, where the inverse of $f(x) \in L$ is computed via the extended Euclid's algorithm for $f(\lambda)$ and $g(\lambda)^s$). For every $u \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times$ (one representative from each class), set $b_u = \rho(u)b\rho(u)^{-1}$.
- (4) The matrices $\{b_u\}$ generate a subgroup G of $\mathrm{PGL}_d(L)$, which contains $\mathrm{PSL}_d(L)$. The index $r = [G : \mathrm{PSL}_d(L)]$ can be computed using Proposition 6.7.
- (5) Let $P = \{(u_1, \dots, u_d) : b_{u_1} \dots b_{u_d} = 1\}$. For every $k = 1, \dots, d-1$, let S_k denote the set of products $b_{u_1} \dots b_{u_k}$ for the headers (u_1, \dots, u_k) of vectors in P . There will be $\binom{d}{k}_q$ different products (up to scalar multiples over L), and S_{d-k} is the set of inverses of S_k .
- (6) The Cayley complex of G with respect to $S_1 \cup \dots \cup S_{d-1}$ defines a simplicial complex (the clique complex of the Cayley graph). This complex is Ramanujan by Theorem 6.2.

10. AN EXAMPLE

We demonstrate the construction of the group Γ , and the embedding into $d^2 \times d^2$ and $d \times d$ matrices. We choose $q = 2$ and $d = 3$, and write $\mathbb{F}_8 = \mathbb{F}_2[v \mid v^3 = v + 1]$. Fix the basis $\{\zeta_0, \zeta_1, \zeta_2\} = \{1, v, v^2\}$ for \mathbb{F}_8 over \mathbb{F}_2 . Then $\{\zeta_i z^j\}$ is a basis for $\mathcal{A}(R) = R[v, z \mid v^3 = v + 1, z^3 = 1 + y, zvz^{-1} = v^2]$ over $R = \mathbb{F}_2[y, 1/y, 1/(1+y)]$. By definition, $\Gamma \subseteq \mathcal{A}(R)^\times$ is generated by the conjugates $u(1-z^{-1})u^{-1}$ for $u \in \mathbb{F}_8^\times/\mathbb{F}_2^\times = \mathbb{F}_8^\times = \langle v \rangle$. These generators act on $\mathcal{A}(R)$ by conjugation, and the corresponding 9×9 matrices over R are given in the right-hand column of Table 9 (they are all of the form $a + by^{-1}$ for $a, b \in \text{GL}_9(\mathbb{F}_2)$).

The relations of Γ were discussed in Section 5. If we set $b_i = v^i(1 - z^{-1})v^{-i}$, conjugation by v induces an outer automorphism of order 7 (namely $b_i \mapsto b_{i+1}$), and the defining relations are

$$b_0 b_3 = b_4 b_2 = b_6 b_5,$$

$$b_0 b_5 = b_2 b_1 = b_3 b_6,$$

$$b_0 b_6 = b_1 b_4 = b_5 b_3,$$

$$b_0 b_3 b_1 = 1,$$

and their v -conjugates.

We continue to find a representation of dimension 3 of a finite quotient of Γ . The first step is to define R as a subring of $\bar{R} = R[x]$, so we need to choose β with $\text{tr}(\beta) \neq 0$; here we take $\beta = 1 + v$, so that $1 + y = N_{\mathbb{F}_8/\mathbb{F}_2}(1 + \beta x) = 1 + x + x^3$ and $y(x) = x + x^3$. Then $R \hookrightarrow \bar{R}$ by sending $y \mapsto y(x)$.

Mimicking Example 8.3, the embedding $\mathcal{A}(R) \hookrightarrow \text{PGL}_3(\bar{R})$ is defined by

$$v \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\begin{aligned} z &\mapsto \rho(1 + \beta x) \cdot (1 \otimes \phi) \\ &= \begin{pmatrix} 1+x & 0 & x \\ x & 1+x & x \\ 0 & x & 1+x \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+x & x & x \\ x & x & 1 \\ 0 & 1+x & 1 \end{pmatrix}. \end{aligned}$$

Check that z^3 is mapped to the scalar matrix $1 + x + x^3 = 1 + y(x)$, so

$$z^{-1} = \frac{1}{1+y(x)} z^2, \text{ and } b = 1 - z^{-1} \mapsto \frac{1}{1+y(x)} \begin{pmatrix} x + x^3 & x^2 & x + x^2 \\ x & x^3 & 1 + x + x^2 \\ x + x^2 & 1 + x^2 & 1 + x^3 \end{pmatrix},$$

where $1 + y(x)$ is invertible in \bar{R} by definition. We emphasize that this embedding is not onto (since $\mathcal{A}(R)$ does not split).

Finally, we present a finite quotient of Γ . Choose $e = 4$, so Γ will map onto $\mathrm{PGL}_3(L)$, where L is a finite local ring whose residue field is \mathbb{F}_{16} (in general Γ could be mapped onto a subgroup of $\mathrm{PGL}_3(L)$ containing $\mathrm{PSL}_3(L)$). Applying the proof of Theorem 9.1 and the algorithm given in Section 9, we look for an element $\alpha \in \mathbb{F}_{16}$ such that $y(\alpha)$ is a generator.

Arbitrarily choosing an irreducible polynomial of degree 4 over \mathbb{F}_2 , we set $\mathbb{F}_{16} = \mathbb{F}_2[t \mid t^4 + t + 1 = 0]$. It then turns out that $y(t) = t^3 + t$ is a generator (since $y(t^2) \neq y(t)$), so we choose $\alpha = t$ and $\gamma = y(t)$. The minimal polynomial of α is of course $g(\lambda) = \lambda^4 + \lambda + 1$, and the minimal polynomial of γ is computed to be $p(\lambda) = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$. Now $g(\lambda)$ divides $p(y(\lambda)) = p(\lambda^3 + \lambda) = \lambda^{12} + \lambda^9 + \lambda^7 + \lambda^6 + \lambda^5 + \lambda^4 + \lambda^2 + \lambda + 1 = g(\lambda)(\lambda^8 + \lambda^4 + \lambda^3 + \lambda^2 + 1)$. For any $s \geq 1$, when we take $I = \langle p(y)^s \rangle$ and $\bar{I} = \langle g(x)^s \rangle$, we have $R \cap \bar{I} = I$. Since $\deg(p) = \deg(g)$, $R/I \cong \bar{R}/\bar{I} = \mathbb{F}_2[x]/\langle (x^4 + x + 1)^s \rangle$.

The index over PSL_d is the order of $y/(1 + y) = y^3 + y + 1$ in $\mathbb{F}_2[y]/\langle y^4 + y^3 + y^2 + y + 1 \rangle$ modulo its cubes; the order is 3, so Γ maps onto $\mathrm{PGL}_d(L)$. The 3×3 matrices over \bar{R} corresponding to the generators b_i of $\Gamma/(G'(R, I) \cap \Gamma) \cong \mathrm{PGL}_3(R/I)$ are given in the intermediate column of Table 9. The Cayley graph with respect to these generators is Ramanujan by Theorem 6.2.

11. GLOSSARY

$\mathcal{A}(S)$ — an S -algebra for any R_T -algebra S with the relations given in Equation (4) (see Section 3)

$\mathcal{B} = \mathcal{B}_d(F)$ — Bruhat-Tits building $= \mathrm{PGL}_d(F)/\mathrm{PGL}_d(\mathcal{O}) = G(F)/K$

\mathcal{B}^0 — the set of vertices of \mathcal{B}

\mathcal{B}^i — the i -skeleton of \mathcal{B}

$\mathcal{B}_I = \Gamma(I) \backslash G(F)/K \cong \Gamma/\Gamma(I)$

d — an integer ≥ 1

$F = \mathbb{F}_q((y)) = k_y$ (see Figure 1)

$\tilde{G}'(S) = \mathcal{A}(S)^\times$ (in particular $\tilde{G}'(R) = \mathcal{A}(R)^\times$, and likewise for the other groups defined over R or other R_T -algebras)

$G'(S) = \mathcal{A}(S)^\times/S^\times$.

$\tilde{G}'_1(S)$ — elements of $\tilde{G}'(S)$ of norm 1

$G'_1(S)$ — the image of $\tilde{G}'_1(S)$ under the map $\tilde{G}'(S) \rightarrow G'(S)$ (see Figure 5)

$\tilde{G}(S) = \mathrm{GL}_d(S)$

$G(S) = \mathrm{PGL}_d(S)$

[illegible]

FIGURE 9. Generators of Γ in the example of Section 10

$$\tilde{G}_1(S) = \mathrm{SL}_d(S)$$

$$G_1(S) = \mathrm{PSL}_d(S)$$

$\tilde{G}'(S, I)$ — congruence subgroup of $\tilde{G}'(S)$; kernel of $\tilde{G}'(S) \rightarrow \tilde{G}'(S/I)$

$G'(S, I)$ — kernel of $G'(S) \rightarrow G'(S/I)$ (for other congruence subgroups, see Section 6 and in particular Figure 5)

$\tilde{\Gamma} = \langle b_u : u \in \mathbb{F}_{q^d}^\times / \mathbb{F}_q^\times \rangle$ (see Section 4 for definition of $b_u \in \mathcal{A}(R)$)

Γ — the image of $\tilde{\Gamma}$ under $\tilde{G}'(R) \rightarrow G'(R)$

$\Gamma(I) = \Gamma \cap G'(R, I)$

$\Gamma_1 = \Gamma \cap G'_1(R)$ (see Figure 6)

$H = G'(R, I)$

$H_1 = H \cap G'_1(R)$

$I = \langle p^s \rangle \triangleleft R$, $p \in \mathbb{F}_q[y]$ is a prime and $s \geq 1$

$k = \mathbb{F}_q(y)$ (see Figure 1)

$k_1 = \mathbb{F}_{q^d}(y)$ (see Figure 1)

k_p — the completion of k with respect to p

$\bar{k} = \mathbb{F}_q(x)$

$L = R/I$, a finite local ring

$L_0 = R/J = R/\langle p \rangle$, the residue field of L

$\bar{L} = \bar{R}/\bar{I}$

$\mathcal{O} = \mathbb{F}_q[[y]]$ — the ring of integers of F

\mathcal{O}_p — the ring of integers of k_p

q — a fixed prime power, the order of the finite field \mathbb{F}_q

$R_T = \mathbb{F}_q[y, 1/(1+y)] \subseteq \mathcal{O}$

$R = \mathbb{F}_q[y, 1/y, 1/(1+y)]$

$R_0 = \mathbb{F}_q[1/y]$

$\bar{R} = R[x]$

$\bar{R}_T = R_T[x]$

$\varrho : \mathcal{B}^0 \rightarrow \mathbb{Z}/d\mathbb{Z}$ — the color function of the building

ν_p — valuation of k for the prime polynomial $p \in \mathbb{F}_q[y]$

$\nu_{1/y}$ — the minus degree valuation of k

y — a transcendental variable over \mathbb{F}_q , generates k

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